AIRCRAFT EQUATIONS

of

MOTION

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DEFINITION OF SYMBOLS

SYMBOL	DESCRIPTION	DIMENSION
Absolute	Designates a vector quantity measured in inertial axes system or the absolute value of a number.	
A_{X_A}, A_{Z_A}	Coordinates in body axes system of origin of stability axes. Origin of stability axes is presumed to lie in X_A, Y_A plane. A_{X_A} and A_{Z_A} are respectively positive in the positive X_A and Z_A directions.	(ft)
B_{X_A}, B_{Y_A}, B_{Z_A}	Coordinates in body axes system of origin of thrust vector. B_{X_A}, B_{Y_A}, B_{Z_A} respectively positive in positive X_A, Y_A, Z_A body axes directions.	(ft)
b	Wing span	(ft)
C_D	Total aerodynamic drag coefficient. Coefficient of projection on X_S stability axis of total aerodynamic force. $$C_D = \frac{-F_{X_S}}{\left(\frac{\rho V_p^2}{2}\right) S}$$	(non-dimensional)
C_L	Total aerodynamic lift coefficient. Coefficient of projection on Z_S, stability axis of total aerodynamic force. $$C_L = \frac{-F_{Z_S}}{\left(\frac{\rho V_p^2}{2}\right) S}$$	(non-dimensional)
C_1	Total aerodynamic rolling moment coefficient. Coefficient of projection on X_S stability axis of total aerodynamic moment. $$C_1 = \frac{M_{X_S}}{\left(\frac{\rho V_p^2}{2}\right) Sb}$$	(non-dimensional)

SYMBOL	DESCRIPTION	DIMENSION
C_m	Total aerodynamic pitching moment coefficient. Coefficient of projection on Y_S stability axis of total aerodynamic moment. $$C_m = \frac{M_{Y_S}}{\left(\frac{\rho V_p^2}{2}\right) Sc}$$	(non-dimensional)
C_n	Total aerodynamic yawing moment coefficient. Coefficient of projection on Z_S stability axis of total aerodynamic moment. $$C_n = \frac{M_{Z_S}}{\left(\frac{\rho V_p^2}{2}\right) Sb}$$	(non-dimensional)
C_y	Total aerodynamic side force coefficient. Coefficient of projection on Y_S stability axis of total aerodynamic force. $$C_{Y_S} = \frac{F_{Y_S}}{\left(\frac{\rho V_p^2}{2}\right) S}$$	(non-dimensional)
c	Mean aerodynamic chord	(ft)
E_{X_W}	Projection of total applied force vector on X_W wind axis. Positive in the positive X_W axis direction.	(lbs)
E_{Y_W}	Projection of total applied force vector on the Y_W wind axis. Positive in the positive Y_W wind axis direction.	(lbs)
E_{Z_W}	Projection of total applied force vector on the Z_W wind axis. Positive in the positive Z_W wind axis direction.	(lbs)
g	Gravitational constant	32.2 ft/sec^2
$\bar{i}$	Unit vector in positive X_A axis direction.	(non-dimensional)
$\bar{j}$	Unit vector in positive Y_A axis direction.	(non-dimensional)

SYMBOL	DESCRIPTION	DIMENSION
$\overline{k}$	Unit vector in positive Z_A axis direction	(non-dimensional)
I_e	Moment of inertia of rotating engine parts about engine spin axis.	$(lbs)(ft)(sec)^2$
I_{XX}	Moment of inertia of the aircraft about the X_A body axis. $I_{XX} = \sum_1^i m_i \left[{Y_A}_i^2 + {Z_A}_i^2 \right]$ where "i" represents the generic mass particle of the aircraft and m_i mass of the generic particle.	$(lbs)(ft)(sec)^2$
I_{YY}	Moment of inertia of the aircraft about the Y_A body axis. $I_{YY} = \sum_1^i m_i \left[{X_A}_i^2 + {Z_A}_i^2 \right]$	$(lbs)(ft)(sec)^2$
I_{ZZ}	Moment of inertia of the aircraft about the Z_A body axis. $I_{ZZ} = \sum_1^i m_i \left[{X_A}_i^2 + {Y_A}_i^2 \right]$	$(lbs)(ft)(sec)^2$
J_{XZ}	Product of inertia due to non-symmetric mass distribution with respect to the X_A, Y_A body axes plane. $J_{XZ} = \sum_1^i m_i \left[{X_A}_i \, {Z_A}_i \right]$	$(lbs)(ft)(Sec)^2$
J_{XY}	Product of inertia due to non-symmetric mass distribution with respect to X_A, Z_A body axes plane. $J_{XY} = \sum_1^i m_i \left[{X_A}_i \, {Y_A}_i \right]$	$(lbs)(ft)(sec)^2$
J_{YZ}	Product of inertia due to non-symmetric mass distribution with respect to the X_A, Z_A body axes plane. $J_{YZ} = \sum_1^i m_i \left[{Y_A}_i \, {Z_A}_i \right]$	$(lbs)(ft)(sec)^2$

SYMBOL	DESCRIPTION	DIMENSION
M_{X_A}	Projection of total applied moment on the X_A body axis. Positive in the positive X_A axis direction.	(lbs)(ft)
M_{Y_A}	Projection of total applied moment on the Y_A body axis. Positive in the positive Y_A body axis direction.	(lbs)(ft)
M_{Z_A}	Projection of total applied moment on the Z_A body axis. Positive in the positive Z_A body axis direction.	(lbs)(ft)
p_A	Projection on the X_A body axis of the body axes system absolute rotational velocity vector. Positive in the positive X_A body axis direction.	(radians/sec)
$\dot{p}_A$	Time rate of change of the projection on the X_A body axis of the absolute rotational velocity vector of the body axes system. Positive for increasingly positive values of p_A.	(radians/sec^2)
q	Dynamic pressure $q = \frac{\rho V_p^2}{2}$	(pounds)
q_A	Projection on Y_A body axis of the body axes system absolute rotational velocity vector. Positive in the positive Y_A body axis direction.	(radians/sec)
$\dot{q}_A$	Time rate of change of the projection on the Y_A body axis of the absolute rotational velocity vector of the body axes system. Positive for increasingly positive values of q_A.	(radians/sec^2)
r_A	Projection on the Z_A body axis of the body axes system absolute rotational velocity vector. Positive in the positive Z_A body axis direction.	(radians/sec)
$\dot{r}_A$	Time rate of change of the projection on the Z_A body axis of the absolute rotational velocity vector of the body axes system. Positive for increasingly positive values of r_A.	(radians/sec^2)
S	Wing area	(ft)^2

SYMBOL	DESCRIPTION	DIMENSION
$\bar{s}_1$	Unit vector in positive X_S axis direction.	(non-dimensional)
$\bar{s}_2$	Unit vector in positive Y_S axis direction.	(non-dimensional)
$\bar{s}_3$	Unit vector in positive Z_S axis direction.	(non-dimensional)
T_{X_A}	Projection of total thrust on X_A body axis. $T_{X_A} = \sum_1^i T_{X_{A_i}}$ Positive in positive X_A body axis direction.	(pounds)
$T_{X_{A_i}}$	Projection on X_A body axis of thrust from "i'th" engine. Positive in positive X_A body axis direction.	(pounds)
T_{Y_A}	Projection of total thrust on Y_A body axis. $T_{Y_A} = \sum_1^i T_{Y_{A_i}}$ Positive in positive Y_A body axis direction.	(pounds)
$T_{Y_{A_i}}$	Projection on Y_A body axis of thrust from "i'th" engine. Positive in positive Y_A body axis direction.	(pounds)
T_{Z_A}	Projection of total thrust on Z_A body axis $T_{Z_A} = \sum_1^i T_{Z_{A_i}}$ Positive in positive Z_A body axis direction.	(pounds)
$T_{Z_{A_i}}$	Projection on Z_A body axis of thrust from "i'th" engine. Positive in positive Z_A body axis direction.	(pounds)
$\bar{s}$	Unit vector in positive X_E axis direction.	(non-dimensional)
$\bar{t}$	Unit vector in positive Y_E axis direftion.	(non-dimensional)

SYMBOL	DESCRIPTION	DIMENSION
τ_{2i}, $i = L, R, N$	Landing gear side force. L, R, N respectively left main, right main and nose gear. Measured in the X_E, Y_E plane and directed along the perpendicular to the trace line in the X_E, Y_E plane of the plane containing the landing gear wheel. τ_{2i} is positive toward the right wing tip of the aircraft. Origin of τ_{2i} is the intersection of landing gear strut line of action with the X_E, Y_E plane.	(pounds)
τ_{3i}, $i = L, R, N$	Landing gear vertical force. L, R, N respectively left main, right main, and nose gear. Measured parallel to Z_E axis. τ_{3i} is positive in the positive Z_E axis direction. Origin of τ_{3i} is intersection of landing gear line of action with X_E, Y_E plane.	(pounds)
ρ	Density of the atmosphere.	$(lbs)(sec)^2/(ft)^4$
ω_e	Magnitude of engine rotational velocity vector. For zero ε_1 and ε_2 positive when vector is in direction of positive X_A body axis.	(radians/sec)
$\dot{\omega}_e$	Time rate of change of magnitude of engine rotational velocity.	$(radians/sec^2)$

SYMBOL	DESCRIPTION	DIMENSION
$\bar{n}$	Unit vector in positive Z_E axis direction.	(non-dimensional)
V	Magnitude absolute translational velocity vector of the aircraft center of gravity.	(ft/sec)
$\dot{V}$	Time rate of change of the magnitude of the absolute translational velocity vector of the aircraft center of gravity. Positive for increasing values of "V".	(ft/sec^2)
V_P	Magnitude true airspeed vector of the aircraft center of gravity.	(ft/sec)
$\dot{V}_P$	Time rate of change of the magnitude of the true airspeed vector of the aircraft center of gravity.	(ft/sec^2)
W	Aircraft Weight	(pounds)
$\bar{w}_1$	Unit vector in positive X_W axis direction	(non-dimensional)
$\bar{w}_2$	Unit vector in positive Y_W axis direction.	(non-dimensional)
$\bar{w}_3$	Unit vector in positive Z_W axis direction.	(non-dimensional)
X_{AM}	Absolute value of X_A coordinate of intersection main gear line of action with X_A, Y_A body axes plane.	(ft)
X_{AN}	Absolute value of the coordinate of intersection of nose gear strut line of action with X_A body axis.	(ft)
Y_{AM}	Absolute value of Y_A coordinate of intersection main gear line of action with X_A, Y_A body axes plane.	(ft)
α	Aerodynamic angle of attack. The angle between the X_A body axis and the projection on the X_A, Z_A body axes plane of the "V_P" vector. α is zero when the "V_P" vector is coincident with X_A body axis. When looking in the negative Y_A body axis direction, clockwise rotations from the zero α position give positive α.	(radians)

SYMBOL	DESCRIPTION	DIMENSION
$\dot{\alpha}$	Time rate of change of the aerodynamic angle of attack. Positive for increasingly positive values of α.	(radians/sec)
β	Aerodynamic side slip angle. The angle between the true airspeed vector "V_P" and the X_A, Z_A body axes plane. β is zero when the "V_P" vector is in the positive X_A body axis half of the X_A, Z_A body axes plane. When looking in the positive Z_W wind axis direction, clockwise rotations of the "V_P" vector from the zero β position give positive β.	(radians)
$\dot{\beta}$	Time rate of change of the aerodynamic sideslip angle. Positive for increasingly positive values of β.	(radians/sec)
ε_1	Angle between X_A body axis and projection on X_A, Y_A plane of engine spin axis. When looking in the negative Y_A axis direction, counterclockwise rotations from the X_A axis give positive ε_1.	(radians)
ε_2	Angle between engine spin axis and X_A, Y_A plane. When looking in the positive Z_A direction, positive rotations from X_A, Y_A plane give positive ε_2.	(radians)
λ_N	Geometric nose wheel angle. λ_N is zero when the plane of the nose wheel is parallel to the aircraft plane of symmetry. When looking in the positive Z_A axis direction, clockwise rotations of λ_N from the zero position are positive.	(radians)
ψ_{NP}	Angle measured in X_E, Y_E plane between projection of X_A axis and trace line of plane containing nosewheel. When looking in the positive Z_E direction, clockwise rotations from projection of the X_A axis give positive ψ_{NP}.	(radians)
ψ_{PS}	Angle measured in X_E, Y_E plane between projection of X_A axis and trace line of X_A, Z_A plane. When looking in the positive Z_E axis direction, clockwise rotations from projection of the X_A axis give positive ψ_{PS}.	(radians)

SYMBOL	DESCRIPTION	DIMENSION
ψ	Body axes system heading angle. Angle between the X_E inertial axis and the projection of the X_A body axis in the X_E, Y_E plane. ψ is zero when the projection of the X_A body axis in the X_E, Y_E plane is parallel to the X_E inertial axis and in the direction of the X_E inertial axis. When looking in the positive Z_E inertial axis direction, clockwise rotations from zero ψ reference give positive ψ. ψ is indeterminate for $\theta = \pm 90°$.	(radians)
θ	Body axes system pitch angle. Angle between the X_A body axis and the X_E, Y_E plane. θ is positive when the projection on the Z_E inertial axis of the X_A body axis is in the negative Z_E axis direction. θ is zero when the X_A body axis is parallel to the X_E, Y_E plane.	(radians)
ϕ	Body axes system roll angle. Measured in the body axes Y_A, Z_A plane as the angle between the positive Y_A body axis and the line of intersection of the X_E, Y_E inertial axes plane and the Y_A, Z_A body axis plane. ϕ is zero when the Y_A body axis is parallel to the X_E, Y_E plane and the projection of the Z_A body axis on the Z_E inertial axis in the positive Z_E axis direction. ϕ is 180° when the Y_A body axis is parallel to the X_E, Y_E plane and the projection of the Z_A body axis on the Z_E inertial axis in the negative Z_E axis direction. When looking in the positive X_A body axes direction, clockwise rotations from the zero ϕ reference give positive ϕ. ϕ is indeterminate for $\theta = \pm 90°$.	(radians)
τ_{1i}, $i = L, R, N$	Landing gear tangential force. L, R, N respectively left main, right main and nose gear. Measured in the X_E, Y_E plane and directed along the trace line in the X_e, Y_e plane of the plane containing the landing gear wheel. τ_{1i} is positive toward the nose of the aircraft. Origin of τ_{1i} is the intersection of landing gear strut line of action with X_E, Y_E plane.	(pounds)

Derivation of the General Aircraft Equations of Motion

I. Assumptions

The following assumptions are used:

1. The aircraft has constant mass.
2. The aircraft is a rigid body.
3. The air mass in which the aircraft is flying is stationary with respect to the inertial axes system.

(Note: Additional assumptions are made in applying results to OFT, e.g. inertial system fixed in earth.)

II. Axes Systems

Four axes systems are employed; inertial, wind, stability, and body axes.

1. Inertial Axes System

A right-handed triad of mutually perpendicular axes fixed in inertial space. The inertial system is designated by X_E, Y_E, Z_E, with the respective unit vector $\bar{s}$, $\bar{t}$, $\bar{n}$.

The inertial axes system constitutes the inertial frame of reference upon which is based the validity of the application of Newton's Laws of Motion to the problem.

2. Wind Axes System

A right-handed triad of mutually perpendicular axes whose origin is fixed in the aircraft center of gravity, whose "X" axis is coincident with the aircraft velocity vector relative to the air mass in which the aircraft is flying and whose "Z" axis remains in the aircraft plane of symmetry. The wind axes are designated by X_W, Y_W, Z_W, with respective unit vectors $\overline{w_1}$, $\overline{w_2}$, $\overline{w_3}$.

The wind axes are used for the formulation of the aircraft's translational momentum and consequently are the reference axes system for the resulting force equations evolved by the differentiation with respect to time of linear momentum.

Positive X_W is in the direction of the aircraft velocity vector relative to the air mass; positive Y_W is toward the right wing tip, and positive Z_W is toward the bottom of the aircraft.

3. Stability Axes System

A right-handed triad of mutually perpendicular axes whose origin is at some fixed point in the aircraft, usually 25% MAC, whose "X" axis is in the aircraft plane of symmetry and parallel to the projection on the plane of symmetry of the aircraft velocity vector with respect to the air mass and whose "Z" axis is also in the plane of symmetry. The stability axes are designated by X_S, Y_S, Z_S, and the respective unit vectors $\bar{s}_1$, $\bar{s}_2$, $\bar{s}_3$.

The stability axes are used as a reference system for measuring aerodynamic forces and moments.

Positive X_S is toward the nose of the aircraft; positive Y_S is toward the right wing tip, and positive Z_S is toward the floor of the aircraft.

4. Body Axes System

A right-handed triad of mutually perpendicular axes whose origin is fixed at the aircraft center of gravity. Unlike the wind and stability axes whose origins are also tied to the aircraft but whose orientations are keyed to the aircraft velocity vector, the body axes by virtue of assumptions 1 and 2 are completely fixed in the aircraft. The "X" and "Z" body axes are in the plane of symmetry with the "X" body axis parallel to aircraft reference line in the plane of symmetry. The body axes are designated by X_A, Y_A, Z_A, with respective unit vectors $\bar{i}$, $\bar{j}$, $\bar{k}$.

The body axes are used for the formulation of angular momentum and by differentiation with respect to time of this quantity are the reference axes for the moment equations.

Positive X_A is toward the nose of the aircraft; positive Y_A is toward the right wing tip, and positive Z_A is toward the floor of the aircraft.

III. General Aircraft Equations of Motion

1. Force Equations

From the derivation in Appendix 1

$$E_{x_W} = \frac{W}{g}\dot{V} \tag{1}$$

$$E_{y_W} = \frac{W}{g} V r_W \tag{2}$$

$$E_{z_W} = -\frac{W}{g} V q_W \tag{3}$$

From the derivation in Appendix 3

$$r_W = \left[\dot{\beta} - p_A \sin\alpha + r_A \cos\alpha\right] \tag{4}$$

$$q_W = \left[-\dot{\alpha}\cos\beta - p_A \cos\alpha \sin\beta + q_A \cos\beta - r_A \sin\alpha \sin\beta\right] \tag{5}$$

Therefore the force equations become

$$E_{x_W} = \frac{W}{g}\dot{V} \tag{1}$$

$$E_{y_W} = \frac{WV}{g}\left[\dot{\beta} - p_A \sin\alpha + r_A \cos\alpha\right] \tag{2}$$

$$E_{z_W} = -\frac{WV}{g}\left[-\dot{\alpha}\cos\beta - p_A \cos\alpha \sin\beta + q_A \cos\beta - r_A \sin\alpha \sin\beta\right] \tag{3}$$

2. Moment Equations

From the derivation in Appendix 2

$$M_{x_A} = \dot{p}_A I_{xx} + q_A r_A (I_{zz} - I_{yy}) + (r_A p_A - \dot{q}_A) J_{xy} - (p_A q_A + \dot{r}_A) J_{zx} + (r_A^2 - q_A^2) J_{yz} \qquad (8)$$

$$M_{y_A} = \dot{q}_A I_{yy} + r_A p_A (I_{xx} - I_{zz}) + (p_A q_A - \dot{r}_A) J_{yz} - (q_A r_A + \dot{p}_A) J_{xy} + (p_A^2 - r_A^2) J_{zx} \qquad (9)$$

$$M_{z_A} = \dot{r}_A I_{zz} + p_A q_A (I_{yy} - I_{xx}) + (q_A r_A - \dot{p}_A) J_{zx} - (r_A p_A + \dot{q}_A) J_{yz} + (q_A^2 - p_A^2) J_{xy} \qquad (10)$$

$$\dot{V} = \frac{g}{W} E_{x_W}$$

$$\dot{\beta} = \frac{g}{WV} E_{y_W} + p_A \sin\alpha - r_A \cos\alpha$$

$$\dot{\alpha} = \frac{g}{WV\cos\beta} E_{Z_W} - \frac{1}{\cos\beta}\left[p_A \cos\alpha \sin\beta - q_A \cos\beta + r_A \sin\alpha \sin\beta\right]$$

$$\dot{p}_A = \frac{M_{x_A}}{I_{xx}} + \frac{(I_{yy} - I_{ZZ})}{I_{xx}} q_A r_A + \frac{J_{xy}}{I_{xx}}(\dot{q}_A - r_A p_A) + \frac{J_{xz}}{I_{xx}}(\dot{r}_A + p_A q_A) + \frac{J_{yz}}{I_{xx}}(q_A^2 - r_A^2)$$

$$\dot{q}_A = \frac{M_{y_A}}{I_{yy}} + \frac{(I_{ZZ} - I_{xx})}{I_{yy}} r_A p_A + \frac{J_{yz}}{I_{yy}}(\dot{r}_A - p_A q_A) + \frac{J_{xy}}{I_{yy}}(\dot{p}_A + q_A r_A) + \frac{J_{xz}}{I_{yy}}(r_A^2 - p_A^2)$$

$$\dot{r}_A = \frac{M_{Z_A}}{I_{ZZ}} + \frac{(I_{xx} - I_{yy})}{I_{ZZ}} p_A q_A + \frac{J_{xz}}{I_{ZZ}}(\dot{p}_A - q_A r_A) + \frac{J_{yz}}{I_{ZZ}}(\dot{q}_A + r_A p_A) + \frac{J_{xy}}{I_{ZZ}}(p_A^2 - q_A^2)$$

IV. General Applied Forces:

In general, the external forces applied to the aircraft can be broken down into four contributions; aerodynamic, thrust, weight, and forces due to contact (non-catastrophic of course) with the ground.

Therefore, letting $\bar{E}$ represent the total external force vector

$$\bar{E} = \bar{F} + \bar{T} + \bar{W} + \bar{\mathcal{T}}$$

where $\bar{F}$ denotes vector sum of aero forces

$\bar{T}$ denotes vector sum of thrust forces

$\bar{W}$ denotes the aircraft weight vector

$\bar{\mathcal{T}}$ denotes the vector sum of ground reaction forces

Projecting $\bar{E}$ on the wind axes

$$\bar{E} = (E_{x_W})\bar{w}_1 + (E_{y_W})\bar{w}_2 + (E_{Z_W})\bar{w}_3$$

where

$$E_{x_W} = F_{x_W} + T_{x_W} + W_{x_W} + \mathcal{T}_{x_W}$$

$$E_{y_W} = F_{y_W} + T_{y_W} + W_{y_W} + \mathcal{T}_{y_W}$$

$$E_{Z_W} = F_{Z_W} + T_{Z_W} + W_{Z_W} + \mathcal{T}_{Z_W}$$

From Appendix 5

$$F_{xW} = + V_p^2 \frac{\rho S}{2}\left[C_y \sin\beta - C_D \cos\beta\right]$$

$$F_{yW} = + V_p^2 \frac{\rho S}{2}\left[C_y \cos\beta + C_D \sin\beta\right]$$

$$F_{zW} = - V_p^2 \frac{\rho S}{2}\left[C_L\right]$$

From Appendix 6

$$T_{xW} = \sum_{n=1}^{f}\left[T_{xA}\cos\alpha\cos\beta + T_{yA}\sin\beta + T_{zA}\sin\alpha\cos\beta\right]_n$$

$$T_{yW} = \sum_{n=1}^{f}\left[-T_{xA}\cos\alpha\sin\beta + T_{yA}\cos\beta - T_{zA}\sin\alpha\sin\beta\right]_n$$

$$T_{zW} = \sum_{n=1}^{f}\left[-T_{xA}\sin\alpha + T_{zA}\cos\alpha\right]_n$$

f = NUMBER OF ENGINES

From Appendix 7

$$W_{xW} = W\left[\cos\theta\cos\phi\sin\alpha\cos\beta + \cos\theta\sin\phi\sin\beta - \sin\theta\cos\alpha\cos\beta\right]$$

$$W_{yW} = W\left[\sin\theta\cos\alpha\sin\beta + \cos\theta\sin\phi\cos\beta - \cos\theta\cos\phi\sin\alpha\sin\beta\right]$$

$$W_{zW} = W\left[\cos\theta\cos\phi\cos\alpha + \sin\theta\sin\alpha\right]$$

FROM APPENDIX 8

$$T_{XW} = \Big[(T_{1L} + T_{1R})(\cos\Theta \cos\Psi_{PS}) - (T_{2L} + T_{2R})(\cos\Theta \sin\Psi_{PS}) - (T_{3L} + T_{3R} + T_{3N})(\sin\Theta) + (\ldots$$

$$+\Big[(T_{1L} + T_{1R})(\sin\Theta \sin\phi \cos\Psi_{PS} + \cos\phi \sin\Psi_{PS}) + (T_{2L} + T_{2R})(\cos\phi \cos\Psi_{PS} - \sin\Theta \sin \ldots$$

$$+ (T_{1N})(\sin\Theta \ldots$$

$$+\Big[(T_{1L} + T_{1R})(\sin\Theta \cos\phi \cos\Psi_{PS} - \sin\phi \sin\Psi_{PS}) - (T_{2L} + T_{2R})(\sin\phi \cos\Psi_{PS} + \sin\Theta \ldots$$

$$+ (T_{1N})(\sin\Theta \cos \ldots$$

$$T_{YW} = \Big[(T_{1L} + T_{1R})(\sin\Theta \sin\phi \cos\Psi_{PS} + \cos\phi \sin\Psi_{PS}) + (T_{2L} + T_{2R})(\cos\phi \cos\Psi_{PS} - \sin\Theta \ldots$$

$$+ (T_{1N})(\sin\Theta \sin \ldots$$

$$-\Big[(T_{1L} + T_{1R})(\sin\Theta \cos\phi \cos\Psi_{PS} - \sin\phi \sin\Psi_{PS}) - (T_{2L} + T_{2R})(\sin\phi \cos\Psi_{PS} + \sin\Theta \cos\phi \sin \ldots$$

$$+ (T_{1N})(\sin\Theta \cos \ldots$$

$$-\Big[(T_{1L} + T_{1R})(\cos\Theta \cos\Psi_{PS}) - (T_{2L} + T_{2R})(\cos\Theta \sin\Psi_{PS}) - (T_{3L} + T_{3R} + T_{3N})\sin\Theta + (T_{1N}) \ldots$$

$$T_{ZW} = \Big[(T_{1L} + T_{1R})(\sin\Theta \cos\phi \cos\Psi_{PS} - \sin\phi \sin\Psi_{PS}) - (T_{2L} + T_{2R})(\sin\phi \cos\Psi_{PS} + \sin\Theta \cos\phi \ldots$$

$$+ (T_{1N})(\sin\Theta \cos \ldots$$

$$-\Big[(T_{1L} + T_{1R})(\cos\Theta \cos\Psi_{PS}) - (T_{2L} + T_{2R})(\cos\Theta \sin\Psi_{PS}) - (T_{3L} + T_{3R} + T_{3N})(\sin\Theta) + (T_{1N})(\cos\Theta \ldots$$

WHERE: $\tan\Psi_{PS} = \Big[-\tan\phi \sin\Theta\Big]$

$$\tan\Psi_{NP} = \Big[-\tan\phi \sin\Theta + \frac{\tan\lambda_N \cos\Theta}{\cos\phi}\Big]$$

$$T_{1N})(\cos\Theta \cos\Psi_{NP}) - (T_{2N})(\cos\Theta \sin\Psi_{NP})\Big]\cos\alpha \cos\beta$$

$$\sin\Psi_{PS}) + (T_{3L} + T_{3R} + T_{3N})(\cos\Theta \sin\phi)$$

$$\sin\phi \cos\Psi_{NP} + \cos\phi \sin\Psi_{NP}) + (T_{2N})(\cos\phi \cos\Psi_{NP} - \sin\Theta \sin\phi \sin\Psi_{NP})\Big]\sin\beta$$

$$\cos\phi \sin\Psi_{PS}) + (T_{3L} + T_{3R} + T_{3N})(\cos\Theta \cos\phi)$$

$$\phi \cos\Psi_{NP} - \sin\phi \sin\Psi_{NP}) - (T_{2N})(\sin\phi \cos\Psi_{NP} + \sin\Theta \cos\phi \sin\Psi_{NP})\Big]\sin\alpha \cos\beta$$

$$\sin\phi \sin\Psi_{PS}) + (T_{3L} + T_{3R} + T_{3N})(\cos\Theta \sin\phi)$$

$$\cos\Psi_{NP} + \cos\phi \sin\Psi_{NP}) + (T_{2N})(\cos\phi \cos\Psi_{NP} - \sin\Theta \sin\phi \sin\Psi_{NP})\Big]\cos\beta$$

$$\Psi_{PS}) + (T_{3L} + T_{3R} + T_{3N})(\cos\Theta \cos\phi)$$

$$\phi \cos\Psi_{NP} - \sin\phi \sin\Psi_{NP}) - (T_{2N})(\sin\phi \cos\Psi_{NP} + \sin\Theta \cos\phi \sin\Psi_{NP})\Big]\sin\alpha \sin\beta$$

$$\cos\Theta \cos\Psi_{NP}) - (T_{2N})(\cos\Theta \sin\Psi_{NP})\Big]\cos\alpha \sin\beta$$

$$\sin\Psi_{PS}) + (T_{3L} + T_{3R} + T_{3N})(\cos\Theta \cos\phi)$$

$$\cos\Psi_{NP} - \sin\phi \sin\Psi_{NP}) - (T_{2N})(\sin\phi \cos\Psi_{NP} + \sin\Theta \cos\phi \sin\Psi_{NP})\Big]\cos\alpha$$

$$\Psi_{NP}) - (T_{2N})(\cos\Theta \sin\Psi_{NP})\Big]\sin\alpha$$

COMBINING THE VARIOUS FORCE CONTRIB

$$E_{xW} = +V_P^2 \frac{\rho S}{2}\left[C_y \sin\beta - C_D \cos\beta\right] + \sum_{n=1}^{f}\left[T_{xA}\cos\alpha\cos\beta + T_{yA}\sin\beta\right.$$

$$+ \left(T_{1L} + T_{1R}\right)\left[\left(\cos\Theta\cos\Psi_{PS}\cos\alpha\cos\beta\right) + \left(\sin\Theta\sin\phi\cos\Psi_{PS} + \cos\phi\right.\right.$$

$$-\left(T_{2L} + T_{2R}\right)\left[\left(\cos\Theta\sin\Psi_{PS}\cos\alpha\cos\beta\right) + \left(\sin\Theta\sin\phi\sin\Psi_{PS} - \cos\phi\right.\right.$$

$$-\left(T_{3L} + T_{3R} + T_{3N}\right)\left[\left(\sin\Theta\cos\alpha\cos\beta\right) - \left(\cos\Theta\sin\phi\sin\beta\right) - \left(\cos\Theta\, c\right.\right.$$

$$+\left(T_{1N}\right)\left[\left(\cos\Theta\cos\Psi_{NP}\cos\alpha\cos\beta\right) + \left(\sin\Theta\sin\phi\cos\Psi_{NP} + \cos\phi\sin\right.\right.$$

$$-\left(T_{2N}\right)\left[\left(\cos\Theta\sin\Psi_{NP}\cos\alpha\cos\beta\right) + \left(\sin\Theta\sin\phi\sin\Psi_{NP} - \cos\phi\cos\Psi_{N}\right.\right.$$

WHERE: $\tan\Psi_{PS} = \left[-\tan\phi\sin\Theta\right]$

$$\tan\Psi_{NP} = \left[-\tan\phi\sin\Theta + \frac{\tan\lambda_N\cos\Theta}{\cos\phi}\right]$$

f = NUMBER OF ENGINES

UTIONS GIVES:

$+ T_{Z_A} \sin\alpha \cos\beta \Big]_n + W\left[\cos\Theta\cos\phi\sin\alpha\cos\beta + \cos\Theta\sin\phi\sin\beta - \sin\Theta\cos\alpha\cos\beta\right]$

$\sin\psi_{PS}) \sin\beta + (\sin\Theta\cos\phi\cos\psi_{PS} - \sin\phi\sin\psi_{PS}) \sin\alpha\cos\beta \Big]$

$\cos\psi_{PS}) \sin\beta + (\sin\Theta\cos\phi\sin\psi_{PS} + \sin\phi\cos\psi_{PS}) \sin\alpha\cos\beta \Big]$

$s\phi\sin\alpha\cos\beta)) \Big]$

$\psi_{NP}) \sin\beta + (\sin\Theta\cos\phi\cos\psi_{NP} - \sin\phi\sin\psi_{NP}) \sin\alpha\cos\beta \Big]$

$) \sin\beta + (\sin\Theta\cos\phi\sin\psi_{NP} + \sin\phi\cos\psi_{NP}) \sin\alpha\cos\beta \Big]$

$$E_{yw} = +V_p^2 \frac{\rho S}{2}\left[C_y \cos\beta + C_D \sin\beta\right] + \sum_{n=1}^{f}\left[-T_{x_A}\cos\alpha\sin\beta + T_{y_A}\cos\beta - T_{z_A}\sin\alpha\sin\beta\right]_n$$

$$-(T_{1L}+T_{1R})\left[(\cos\Theta\cos\Psi_{PS}\cos\alpha\sin\beta) - (\sin\Theta\sin\phi\cos\Psi_{PS} + \cos\phi\sin\Psi_{PS})\cos\right.$$

$$+(T_{2L}+T_{2R})\left[(\cos\Theta\sin\Psi_{PS}\cos\alpha\sin\beta) - (\sin\Theta\sin\phi\sin\Psi_{PS} - \cos\phi\cos\Psi_{PS})\cos\right.$$

$$+(T_{3L}+T_{3R}+T_{3N})\left[(\sin\Theta\cos\alpha\sin\beta) + (\cos\Theta\sin\phi\cos\beta) - (\cos\Theta\cos\phi\sin\alpha\right.$$

$$-(T_{1N})\left[(\cos\Theta\cos\Psi_{NP}\cos\alpha\sin\beta) - (\sin\Theta\sin\phi\cos\Psi_{NP} + \cos\phi\sin\Psi_{NP})\cos\beta + (\sin\right.$$

$$+(T_{2N})\left[(\cos\Theta\sin\Psi_{NP}\cos\alpha\sin\beta) - (\sin\Theta\sin\phi\sin\Psi_{NP} - \cos\phi\cos\Psi_{NP})\cos\beta + (\sin\right.$$

WHERE: $\tan\Psi_{PS} = \left[-\tan\phi\sin\Theta\right]$

$$\tan\Psi_{NP} = \left[-\tan\phi\sin\Theta + \frac{\tan\lambda_N\cos\Theta}{\cos\phi}\right]$$

f = NUMBER OF ENGINES

$$+ W\left[\sin\Theta\cos\alpha\sin\beta + \cos\Theta\sin\phi\cos\beta - \cos\Theta\cos\phi\sin\alpha\sin\beta\right]$$

$$\beta + \left(\sin\Theta\cos\phi\cos\psi_{PS} - \sin\phi\sin\psi_{PS}\right)\sin\alpha\sin\beta\Big]$$

$$\beta + \left(\sin\Theta\cos\phi\sin\psi_{PS} + \sin\phi\cos\psi_{PS}\right)\sin\alpha\sin\beta\Big]$$

$$\text{in}\,\beta)\Big]$$

$$\Theta\cos\phi\cos\psi_{NP} - \sin\phi\sin\psi_{NP})\sin\alpha\sin\beta\Big]$$

$$\Theta\cos\phi\sin\psi_{NP} + \sin\phi\cos\psi_{NP})\sin\alpha\sin\beta\Big]$$

$$E_{ZW} = -V_P^2 \frac{\rho S}{2}\left[C_L\right] + \sum_{n=1}^{\#}\left[-T_{X_A}\sin\alpha + T_{Z_A}\cos\alpha\right]_n + W\left[\cos\Theta\cos\phi\cos\alpha + \ldots\right.$$

$$-(T_{1L}+T_{1R})\left[(\cos\Theta\cos\Psi_{PS}\sin\alpha) - (\sin\Theta\cos\phi\cos\Psi_{PS} - \sin\phi\sin\Psi_{PS})\cos\ldots\right.$$

$$+(T_{2L}+T_{2R})\left[(\cos\Theta\sin\Psi_{PS}\sin\alpha) - (\sin\Theta\cos\phi\sin\Psi_{PS} + \sin\phi\cos\Psi_{PS})\cos\alpha\right]$$

$$+(T_{3L}+T_{3R}+T_{3N})\left[(\sin\Theta\sin\alpha) + (\cos\Theta\cos\phi\cos\alpha)\right]$$

$$-(T_{1N})\left[(\cos\Theta\cos\Psi_{NP}\sin\alpha) - (\sin\Theta\cos\phi\cos\Psi_{NP} - \sin\phi\sin\Psi_{NP})\cos\alpha\right]$$

$$+(T_{2N})\left[(\cos\Theta\sin\Psi_{NP}\sin\alpha) - (\sin\Theta\cos\phi\sin\Psi_{NP} + \sin\phi\cos\Psi_{NP})\cos\alpha\right]$$

WHERE: $\tan\Psi_{PS} = \left[-\tan\phi\sin\Theta\right]$

$$\tan\Psi_{NP} = \left[-\tan\phi\sin\Theta \quad \frac{\tan\lambda_N\cos\Theta}{\cos\phi}\right]$$

$\#$ = NUMBER OF ENGINES

$$\ldots \text{IN}\,\theta\ \text{SIN}\,\alpha]$$

$$\ldots \alpha]$$

V. General Applied Moments

In general, the external moments applied to the aircraft can be broken down into five contributions; aerodynamic moments computed with respect to the stability axes, moments due to aerodynamic forces computed with respect to the stability axes, and arising because of non-coincidence between aircraft center of gravity and origin of stability axes, engine gyroscopic effects, moments due to contact with the ground, and moments due to thrust.

Letting $\overline{M}$ represent the total external moment vector

$$\overline{M} = \overline{M}_a + \overline{M}_T + \overline{M}_e + \overline{M}_G$$

where $\overline{M}_a$ denotes vector sum moments due to aerodynamics (pure aerodynamic moments + moments due to aerodynamic forces)

$\overline{M}_T$ denotes vector sum of thrust moments

$\overline{M}_e$ denotes vector sum of engine gyroscopic moments

$\overline{M}_G$ denotes vector sum of ground reaction moments

Projecting $\overline{M}$ on the body axes

$$\overline{M} = (M_{x_A})\bar{i} + (M_{y_A})\bar{j} + (M_{z_A})\bar{k}$$

where

$$M_{x_A} = M_{a_{x_A}} + M_{T_{x_A}} + M_{e_{x_A}} + M_{G_{x_A}}$$

$$M_{y_A} = M_{a_{y_A}} + M_{T_{y_A}} + M_{e_{y_A}} + M_{G_{y_A}}$$

$$M_{z_A} = M_{a_{z_A}} + M_{T_{z_A}} + M_{e_{z_A}} + M_{G_{z_A}}$$

From Appendix 9 and Appendix 10

$$M_{a_{x_A}} = V_p^2 \frac{\rho S}{2}\left[(C_\ell \cos\alpha - C_n \sin\alpha)b - (C_y)A_{z_A}\right]$$

$$M_{a_{y_A}} = V_p^2 \frac{\rho S}{2}\left[(C_m)c + (C_L \sin\alpha - C_D \cos\alpha)A_{z_A} + (C_L \cos\alpha + C_D \sin\alpha)A_{x_A}\right]$$

$$M_{a_{z_A}} = V_p^2 \frac{\rho S}{2}\left[(C_\ell \sin\alpha + C_n \cos\alpha)b + (C_y)A_{x_A}\right]$$

From Appendix 11

$$M_{T_{x_A}} = \left[T_{z_A} B_{y_A} - T_{y_A} B_{z_A}\right]$$

$$M_{T_{y_A}} = \left[T_{x_A} B_{z_A} - T_{z_A} B_{x_A}\right]$$

$$M_{T_{z_A}} = \left[T_{y_A} B_{x_A} - T_{x_A} B_{y_A}\right]$$

From Appendix 12

$$M_{e_{x_A}} = -I_e \dot{\omega}_e\left[\cos\epsilon_1 \cos\epsilon_2\right] + I_e \omega_e\left[(\sin\epsilon_1 \cos\epsilon_2)q_A + (\sin\epsilon_2)r_A\right]$$

$$M_{e_{y_A}} = -I_e \dot{\omega}_e\left[\sin\epsilon_2\right] - I_e \omega_e\left[(\cos\epsilon_1 \cos\epsilon_2)r_A + (\sin\epsilon_1 \cos\epsilon_2)p_A\right]$$

$$M_{e_{z_A}} = +I_e \dot{\omega}_e\left[\sin\epsilon_1 \cos\epsilon_2\right] - I_e \omega_e\left[(\sin\epsilon_2)p_A - (\cos\epsilon_1 \cos\epsilon_2)q_A\right]$$

FROM APPENDIX 8

$$M_{G_{X_A}} = -Y_{AM}\Big[(T_{1L} - T_{1R})(\sin\theta\cos\phi\cos\psi_{PS} - \sin\phi\sin\psi_{PS}) - (T_{2L} - T_{2R})(\sin\theta\cos\phi\sin\psi_{PS}$$

$$-h_L\Big[(T_{1L})(\sin\theta\sin\phi\cos\psi_{PS} + \cos\phi\sin\psi_{PS}) - (T_{2L})(\sin\theta\sin\phi\sin\psi_{PS} - \cos\phi\cos\psi_{P}$$

$$-h_R\Big[(T_{1R})(\sin\theta\sin\phi\cos\psi_{PS} + \cos\phi\sin\psi_{PS}) - (T_{2R})(\sin\theta\sin\phi\sin\psi_{PS} - \cos\phi$$

$$-h_N\Big[(T_{1N})(\sin\theta\sin\phi\cos\psi_{NP} + \cos\phi\sin\psi_{NP}) - (T_{2N})(\sin\theta\sin\phi\sin\psi_{NP} - \cos\phi$$

$$M_{G_{Y_A}} = X_{AM}\Big[(T_{1L} + T_{1R})(\sin\theta\cos\phi\cos\psi_{PS} - \sin\phi\sin\psi_{PS}) - (T_{2L} + T_{2R})(\sin\theta\cos\phi\sin\psi_{PS}$$

$$+\Big[(h_L T_{1L} + h_R T_{1R})(\cos\theta\cos\psi_{PS}) - (h_L T_{2L} + h_R T_{2R})(\cos\theta\sin\psi_{PS}) - (h_L T_{3L} + h_R T_{3R})\sin$$

$$-X_{AN}\Big[(T_{1N})(\sin\theta\cos\phi\cos\psi_{NP} - \sin\phi\sin\psi_{NP}) - (T_{2N})(\sin\theta\cos\phi\sin\psi_{NP} + \sin\phi$$

$$+h_N\Big[(T_{1N})(\cos\theta\cos\psi_{NP}) - (T_{2N})(\cos\theta\sin\psi_{NP}) + (T_{3N})\sin\theta\Big]$$

$$M_{G_{Z_A}} = -X_{AM}\Big[(T_{1L} + T_{1R})(\sin\theta\sin\phi\cos\psi_{PS} + \cos\phi\sin\psi_{PS}) - (T_{2L} + T_{2R})(\sin\theta\sin\phi\sin\psi_{PS}$$

$$+Y_{AM}\Big[(T_{1L} - T_{1R})(\cos\theta\cos\psi_{PS}) - (T_{2L} - T_{2R})(\cos\theta\sin\psi_{PS}) - (T_{3L} - T_{3R})(\sin\theta)$$

$$+X_{AN}\Big[(T_{1N})(\sin\theta\sin\phi\cos\psi_{NP} + \cos\phi\sin\psi_{NP}) - (T_{2N})(\sin\theta\sin\phi\sin\psi_{NP} - \cos\phi\cos$$

$$+ \sin\phi \cos\psi_{PS}) + (T_{3L} - T_{3R})(\cos\theta \cos\phi)\Big]$$

$$) + (T_{3L})(\cos\theta \sin\phi)\Big]$$

$$\cos\psi_{PS}) + (T_{3R})(\cos\theta \sin\phi)\Big]$$

$$\cos\psi_{NP}) + (T_{3N})(\cos\theta \sin\phi)\Big]$$

$$+ \sin\phi \cos\psi_{PS}) + (T_{3L} + T_{3R})(\cos\theta \cos\phi)\Big]$$

$$\theta\Big]$$

$$\cos\psi_{NP}) + (T_{3N})(\cos\theta \cos\phi)\Big]$$

$$- \cos\phi \cos\psi_{PS}) + (T_{3L} + T_{3R})(\cos\theta \sin\phi)\Big]$$

$$\Big]$$

$$\psi_{NP}) + (T_{3N})(\cos\theta \sin\phi)\Big]$$

$$M_{x_A} = V_p^2 \frac{\rho S}{2}\Big[(C_\ell \cos\alpha - C_n \sin\alpha)b - (C_y)A_{Z_A}\Big] + \sum_{n=1}^{\#}\Big[T_{Z_A}B_{y_A} - T_{y_A}B_{Z_A}\Big]_n - \sum_{n=}$$

$$-Y_{AM}\Big[(T_{1L}-T_{1R})(\sin\Theta\cos\phi\cos\psi_{PS} - \sin\phi\sin\psi_{PS}) - (T_{2L}-T_{2R})(\sin\Theta\cos\phi\sin$$

$$-\Big[(h_L T_{1L} + h_R T_{1R})(\sin\Theta\sin\phi\cos\psi_{PS} + \cos\phi\sin\psi_{PS}) - (h_L T_{2L} + h_R T_{2R})(\sin\Theta\sin$$

$$-h_N\Big[(T_{1N})(\sin\Theta\sin\phi\cos\psi_{NP} + \cos\phi\sin\psi_{NP}) - (T_{2N})(\sin\Theta\sin\phi\sin\psi_{NP} - \cos$$

WHERE: $\tan\psi_{PS} = \Big[-\tan\phi\sin\Theta\Big]$

$$\tan\psi_{NP} = \Big[-\tan\phi\sin\Theta + \frac{\tan\lambda_N\cos\Theta}{\cos\phi}\Big]$$

$\#$ = NUMBER OF ENGINES

$$\left[I_e \dot{\omega}_e (\cos\epsilon_1 \cos\epsilon_2) - I_e \omega_e (\sin\epsilon_1 \cos\epsilon_2) q_A - I_e \omega_e (\sin\epsilon_2) r_A\right]_n$$

$$_{S} + \sin\phi \cos\psi_{PS}) + (T_{3L} - T_{3R})(\cos\Theta \cos\phi)\Big]\Big]$$

$$\phi \sin\psi_{PS} - \cos\phi \cos\psi_{PS}) + (h_L T_{3L} + h_R T_{3R})(\cos\Theta \sin\phi)\Big]\Big]$$

$$\phi \cos\psi_{NP}) + (T_{3N})(\cos\Theta \sin\phi)\Big]\Big]$$

$$M_{y_A} = V_p^2 \frac{\rho S}{2}\Big[(C_m)c + (C_L \sin\alpha - C_D \cos\alpha)A_{z_A} + (C_L \cos\alpha + C_D \sin\alpha)A_{x_A}\Big] + \sum_{n=1}^{\#}\Big[T_{x_A} B_{z_A} -$$

$$+ X_{AM}\Big[(T_{1L} + T_{1R})(\sin\Theta\cos\phi\cos\psi_{PS} - \sin\phi\sin\psi_{PS}) - (T_{2L} + T_{2R})(\sin\Theta\cos\phi\sin\psi_{PS} + \sin$$

$$+\Big[(h_L T_{1L} + h_R T_{1R})(\cos\Theta\cos\psi_{PS}) - (h_L T_{2L} + h_R T_{2R})(\cos\Theta\sin\psi_{PS}) - (h_L T_{3L} + h_R T_{3R})(\sin$$

$$- X_{AN}\Big[(T_{1N})(\sin\Theta\cos\phi\cos\psi_{NP} - \sin\phi\sin\psi_{NP}) - (T_{2N})(\sin\Theta\cos\phi\sin\psi_{NP} + \sin\phi\cos\psi$$

$$+ h_N\Big[(T_{1N})(\cos\Theta\cos\psi_{NP}) - (T_{2N})(\cos\Theta\sin\psi_{NP}) + (T_{3N})(\sin\Theta)\Big]$$

WHERE: $\tan\psi_{PS} = \Big[-\tan\phi\sin\Theta\Big]$

$$\tan\psi_{NP} = \Big[-\tan\phi\sin\Theta + \frac{\tan\lambda_N\cos\Theta}{\cos\phi}\Big]$$

$\#$ = NUMBER OF ENGINES

$$T_{Z_A} B_{x_A} \Big]_n - \sum_{n=1}^{4} \Big[I_e \dot{\omega}_e (\sin \epsilon_2) + I_e \omega_e (\cos \epsilon_1 \cos \epsilon_2) r_A + I_e \omega_e (\sin \epsilon_1 \cos \epsilon_2) p_A \Big]_n$$

$$\phi \cos \psi_{PS}) + (T_{3L} + T_{3R})(\cos \Theta \cos \phi) \Big]$$

$$\Theta) \Big]$$

$$_{P}) + (T_{3N})(\cos \Theta \cos \phi) \Big]$$

$$M_{Z_A} = V_P^2 \frac{\rho S}{2}\Big[\big(C_\ell \sin\alpha + C_n \cos\alpha\big)b + \big(C_y\big)A_{x_A}\Big] + \sum_{n=1}^{f}\Big[T_{y_A}B_{x_A} - T_{x_A}B_{y_A}\Big]_n + \sum_{n=1}^{f}\Big[I_e$$

$$- X_{AM}\Big[\big(T_{1L}+T_{1R}\big)\big(\sin\Theta\sin\phi\cos\Psi_{PS} + \cos\phi\sin\Psi_{PS}\big) - \big(T_{2L}+T_{2R}\big)\big(\sin\Theta\sin\phi\, s$$

$$+ Y_{AM}\Big[\big(T_{1L}-T_{1R}\big)\big(\cos\Theta\cos\Psi_{PS}\big) - \big(T_{2L}-T_{2R}\big)\big(\cos\Theta\sin\Psi_{PS}\big) - \big(T_{3L}-T_{3R}\big)\big(\sin\Theta\big)\Big]$$

$$+ X_{AN}\Big[\big(T_{1N}\big)\big(\sin\Theta\sin\phi\cos\Psi_{NP} + \cos\phi\sin\Psi_{NP}\big) - \big(T_{2N}\big)\big(\sin\Theta\sin\phi\sin\Psi_{NP} - c$$

WHERE: $\tan\Psi_{PS} = \Big[-\tan\phi\sin\Theta\Big]$

$$\tan\Psi_{NP} = \Big[-\tan\phi\sin\Theta + \frac{\tan\lambda_N\cos\Theta}{\cos\phi}\Big]$$

f = NUMBER OF ENGINES

$$\dot{\omega}_e(\sin\epsilon_1\cos\epsilon_2) - I_e\,\omega_e(\sin\epsilon_2)p_A + I_e\,\omega_e(\cos\epsilon_1\cos\epsilon_2)q_A\Big]$$

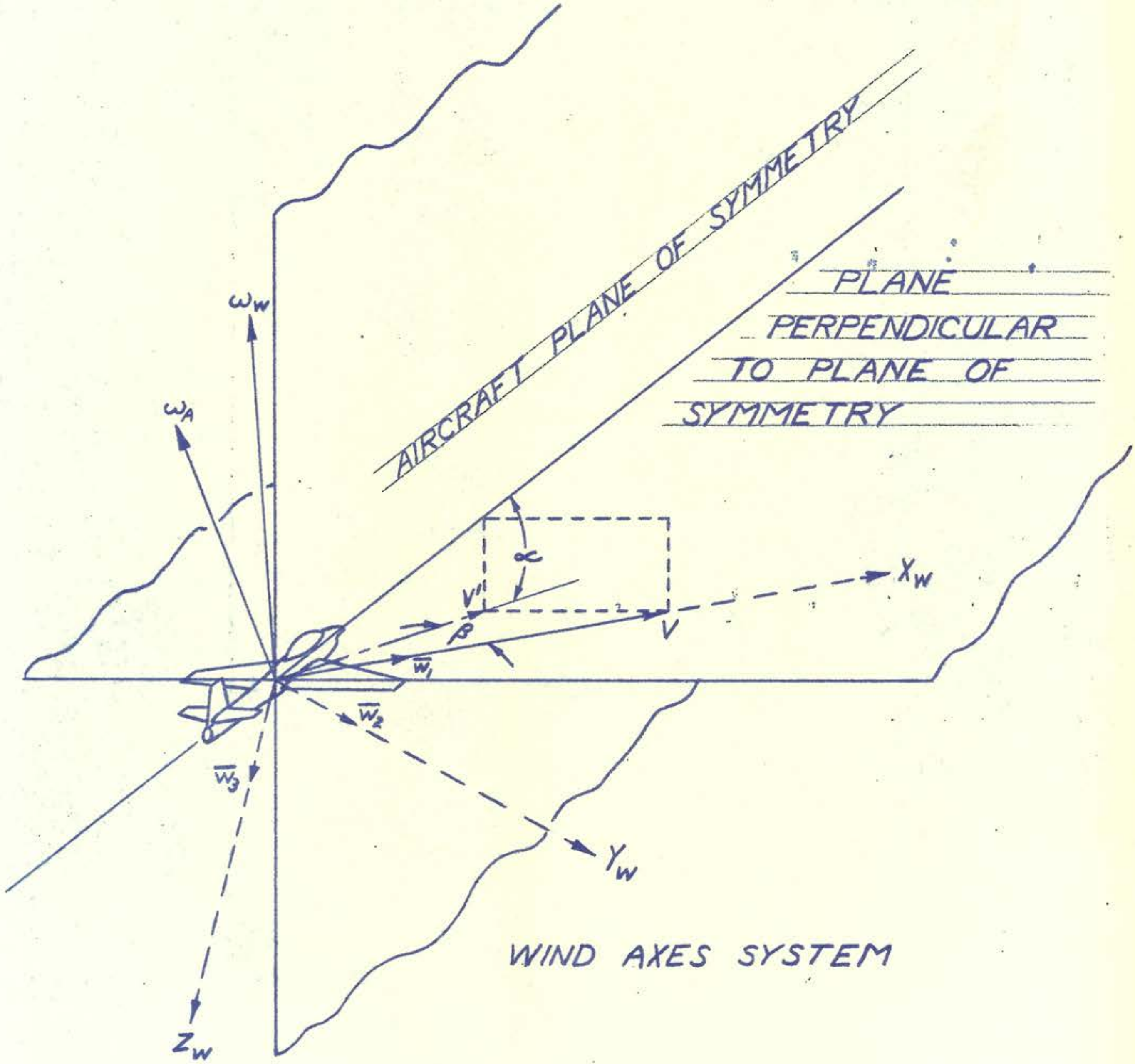
$$\sin\psi_{PS} - \cos\phi\cos\psi_{PS}) + (T_{3L} + T_{3R})(\cos\theta\sin\phi)\Big]$$

$$\cos\phi\cos\psi_{NP}) + (T_{3N})(\cos\theta\sin\phi)\Big]$$

APPENDIX 1

Derivation of Aircraft Equations of Motion

With Respect to Wind Axes



WIND AXES SYSTEM

ω_W = Magnitude of wind axes rotational velocity vector

ω_A = Magnitude of the aircraft rotational velocity vector

V = Magnitude of the aircraft velocity vector

V' = Magnitude of the projection on the plane of symmetry of aircraft velocity vector

α = Angle of attack

β = Side slip angle

Origin of wind axes system is at aircraft center of gravity

X_W = Wind X axis; coincident with aircraft velocity vector

Z_W = Wind Z axis; in plane of symmetry and perpendicular to aircraft velocity vector

Y_W = Wind Y axis; perpendicular to (X_W, Z_W) plane

$\overline{w_1}$ = Unit vector in X_W direction

$\overline{w_2}$ = Unit vector in Y_W direction

$\overline{w_3}$ = Unit vector in Z_W direction

Note: The position of the aircraft relative to the wind axes can change as a function of the angles "α" and "β".

I. Introduction

The equations of motion will be evolved by the application of Newton's Laws

$$\bar{F} = \frac{d}{dt}\left[\bar{Q}\right] \quad \text{AND} \quad \bar{M} = \frac{d}{dt}\left[\bar{H}\right]$$

where $\bar{Q} = \sum m\bar{V}$ is the linear momentum of the aircraft

$\bar{H} = \sum m(\bar{r} \times \bar{V})$ is the moment of momentum of the aircraft

$\bar{V} =$ the total velocity vector of a generic mass particle in the aircraft

$m =$ the mass of a generic mass particle in the aircraft

$\bar{r} =$ the position vector of the generic mass particle "m"; measured with respect to the Wind Axes System.

II. Derivation of force equations with respect to wind axes

For the generic particle "m", the total velocity vector is

$$\bar{V} = V\bar{w}_1 + (\bar{\omega}_A \times \bar{r})$$

where $V\bar{w}_1$ = the velocity vector of the aircraft center of gravity

ω_A = the total rotational velocity vector of the aircraft

The linear momentum with respect to wind axes becomes

$$\bar{Q}_W = \sum m\bar{V} = \sum m\left[V\bar{w}_1 + (\bar{\omega}_A \times \bar{r})\right] = \sum mV\bar{w}_1 + \sum m(\bar{\omega}_A \times \bar{r})$$

The velocity vector of the center of gravity is the same for each particle in the aircraft. In addition, the aircraft rotational velocity vector is the same for each particle in the aircraft. Therefore, in the expression for linear momentum the terms ($V\overline{w_1}$) and (ω_A) can be placed outside the summation signs. This allows $\overline{Q_W}$ to be written as

$$\overline{Q}_W = V\overline{w_1}\left(\sum m\right) + \overline{\omega}_A \times \left(\sum m\overline{r}\right)$$

Now $\sum m$ is the summation of all the mass particles in the aircraft, or

$$\sum m = M = \text{total mass of the aircraft}$$

$\left(\sum m\overline{r}\right)$ is the summation of the vector mass moments with respect to the wind axes system of all the mass particles in the aircraft. Now the wind axes have their origin at the center of gravity of the aircraft. Since the generic mass particle position vector ($\overline{r}$) is measured with respect to the wind axes origin, which is coincident with the aircraft center of gravity, by virtue of the definition of center of gravity

$$\left(\sum m\overline{r}\right) = 0$$

Therefore

$$\overline{Q}_W = MV\overline{w_1}$$

Taking the time derivative of the evolved expression for linear momentum gives

$$\overline{F} = \frac{d}{dt}\left[MV\overline{w_1}\right]$$

<u>Assumption:</u> The mass of the aircraft is considered to be constant.

Therefore

$$\overline{F} = M\frac{d}{dt}\left[V\overline{w_1}\right] = M\left[\frac{dV}{dt}\,\overline{w_1} + V\frac{d\overline{w_1}}{dt}\right]$$

Since the wind axes origin is fixed at the aircraft center of gravity, the wind axes system charges along through space in company with the aircraft; the position of the aircraft relative to the wind axes changes only as a result of changes in α (angle of attack) and β (yaw angle). In short, the wind axes constitute a moving axes system. In the expression

$$\bar{F} = M\left[\frac{dV}{dt}\bar{w}_1 + V\frac{d\bar{w}_1}{dt}\right]$$

the term $\left[\frac{dV}{dt}\bar{w}_1\right]$ takes into account the motion of the origin of the wind axes system and the term $\left[V\frac{d\bar{w}_1}{dt}\right]$ takes into account the rotation of the wind axes system.

Now $$\frac{d}{dt}\bar{w}_1 = \bar{\omega}_w \times \bar{w}_1 = \begin{vmatrix} \bar{w}_1 & \bar{w}_2 & \bar{w}_3 \\ p_w & q_w & r_w \\ 1 & 0 & 0 \end{vmatrix} = r_w\bar{w}_2 - q_w\bar{w}_3$$

where $\bar{\omega}_w = p_w\bar{w}_1 + q_w\bar{w}_2 + r_w\bar{w}_3$

Therefore

$$\bar{F} = M\frac{dV}{dt}\bar{w}_1 + MVr_w\bar{w}_2 - MVq_w\bar{w}_3$$

where q_w and r_w are respectively the <u>magnitudes</u> of the projections on the Y_w and Z_w axes of the wind axes rotation vector $\bar{\omega}_w$.

The term $\bar{F}$ is the resultant applied force vector which can be projected on the wind axes as

$$\bar{F} = E_{x_w}\bar{w}_1 + E_{y_w}\bar{w}_2 + E_{z_w}\bar{w}_3$$

where $E_{x_w}, E_{y_w}, E_{z_w}$ are respectively the <u>magnitudes</u> of the projections of the applied force vector on the X_w, Y_w & Z_w axes.

Therefore

$$E_{x_w}\bar{w}_1 + E_{y_w}\bar{w}_2 + E_{z_w}\bar{w}_3 = M\frac{dV}{dt}\bar{w}_1 + MVr_w\bar{w}_2 - MVq_w\bar{w}_3$$

For the above vector equation to hold, equality must hold between respective components of the left and right hand sides of the equation. Equating components gives rise to the "force equations of motion" written with respect to wind axes

Along the X_w axis $\quad E_{x_w} = M\frac{dV}{dt}$

Along the Y_w axis $\quad E_{y_w} = MVr_w$

Along the Z_w axis $\quad E_{z_w} = -MVq_w$

In summary, $(E_{x_w}, E_{y_w}, E_{z_w})$ are the magnitudes of the projections on the wind axes of the resultant applied force vector. (q_w, r_w) are the magnitudes of the projections on the Y_w and Z_w wind axes of the wind axes rotational velocity vector. (V) is the magnitude of the velocity vector of the aircraft center of gravity. It should be emphasized that the resultant force vector $(\bar{F})$, wind axes rotational velocity vector $(\bar{\omega}_w)$ and the aircraft center of gravity velocity vector $(V\bar{w}_1)$ are measured with respect to fixed space references; it is their projections on the wind axes that pop up in the above equations. Since the wind axes system is set up with the X_w axis coincident with the aircraft center of gravity velocity vector, the only projection of the c.g. velocity vector on the wind axes system is along the X_w axis and has magnitude (V).

(V) is the true airspeed of the aircraft.

III. Derivation of Moment Equations with respect to Wind Axes

Starting with the general expression for moment of momentum

$$\bar{H} = \sum m(\bar{r} \times \bar{V})$$

and the total velocity vector for the generic mass particle " m "

$$\bar{V} = V\bar{W_1} + (\bar{\omega}_A \times \bar{r})$$

the moment of momentum with respect to wind axes becomes

$$\bar{H}_W = \sum m(\bar{r} \times V\bar{W_1}) + \sum m\left[\bar{r} \times (\bar{\omega}_A \times \bar{r})\right]$$

For reasons explained in the derivation of the force equations, the first term on the right side of the above expression is

$$\sum m(\bar{r} \times V\bar{W_1}) = -\sum m(V\bar{W_1} \times \bar{r}) = -V\bar{W_1} \times \sum m\bar{r} = 0$$

$$\text{SINCE } \sum m\bar{r} = 0$$

Therefore $\bar{H}_W = \sum m\left[\bar{r} \times (\bar{\omega}_A \times \bar{r})\right]$

and evaluating the double vector product $\left[\bar{r} \times (\bar{\omega}_A \times \bar{r})\right]$ gives

$$\begin{aligned}\bar{H}_W = \; & \sum m\left[\omega_{xW}(r_{yW}^2 + r_{zW}^2) - \omega_{yW} r_{xW} r_{yW} - \omega_{zW} r_{xW} r_{zW}\right]\bar{W_1} \\ & + \sum m\left[\omega_{yW}(r_{xW}^2 + r_{zW}^2) - \omega_{xW} r_{xW} r_{yW} - \omega_{zW} r_{yW} r_{zW}\right]\bar{W_2} \\ & + \sum m\left[\omega_{zW}(r_{xW}^2 + r_{yW}^2) - \omega_{xW} r_{xW} r_{zW} - \omega_{yW} r_{yW} r_{zW}\right]\bar{W_3}\end{aligned}$$

where (r_{xW}), (r_{yW}) and (r_{zW}) are the <u>magnitudes</u> of the projections on the wind axes of the generic mass particle position <u>vector</u> $(\bar{r})$; ω_{xW}, ω_{yW}, AND ω_{zW} are the magnitudes of the projections on the wind axes of the aircraft rotational velocity vector $(\bar{\omega}_A)$

As measured in the body axes, $(\overline{\omega}_A)$ is given by

$$\overline{\omega}_A = p_A \bar{i} + q_A \bar{j} + r_A \bar{k}$$

Since the transfer equations from body axes to wind axes are

$$\bar{i} = \cos\alpha \cos\beta \; \overline{W}_1 - \cos\alpha \sin\beta \; \overline{W}_2 - \sin\alpha \; \overline{W}_3$$

$$\bar{j} = \sin\beta \; \overline{W}_1 + \cos\beta \; \overline{W}_2$$

$$\bar{k} = \sin\alpha \cos\beta \; \overline{W}_1 - \sin\alpha \sin\beta \; \overline{W}_2 + \cos\alpha \; \overline{W}_3$$

The aircraft rotational velocity vector projected on the wind axes is

$$\overline{\omega}_A = (\omega_{xW}) \overline{W}_1 + (\omega_{yW}) \overline{W}_2 + (\omega_{zW}) \overline{W}_3$$

where

$$\omega_{xW} = \left[p_A \cos\alpha \cos\beta + q_A \sin\beta + r_A \sin\alpha \cos\beta \right]$$

$$\omega_{yW} = \left[-p_A \cos\alpha \sin\beta + q_A \cos\beta - r_A \sin\alpha \sin\beta \right]$$

$$\omega_{zW} = \left[-p_A \sin\alpha + r_A \cos\alpha \right]$$

As indicated in the "force equations" derivation, the aircraft rotational velocity vector ($\overline{\omega_A}$) can be taken outside the summation sign because it does not vary from mass particle to mass particle. The same effect applies to the magnitudes of the projections of the rotation vector ($\overline{\omega_A}$). Therefore

$$\overline{H_W} = \left[\omega_{xW} \sum m\left(r_{yW}^2 + r_{zW}^2\right) - \omega_{yW} \sum m\, r_{xW}\, r_{yW} - \omega_{zW} \sum m\, r_{xW}\, r_{zW}\right]\overline{W_1}$$
$$+\left[\omega_{yW} \sum m\left(r_{xW}^2 + r_{zW}^2\right) - \omega_{xW} \sum m\, r_{xW}\, r_{yW} - \omega_{zW} \sum m\, r_{yW}\, r_{zW}\right]\overline{W_2}$$
$$+\left[\omega_{zW} \sum m\left(r_{xW}^2 + r_{yW}^2\right) - \omega_{xW} \sum m\, r_{xW}\, r_{zW} - \omega_{yW} \sum m\, r_{yW}\, r_{zW}\right]\overline{W_3}$$

The summation quantities in the above expression define the various products and moments of inertia of the aircraft with respect to the wind axes

$$\sum m\, r_{xW}\, r_{yW} = J_{xW yW} \qquad \sum m\left(r_{yW}^2 + r_{zW}^2\right) = I_{xW xW}$$

$$\sum m\, r_{xW}\, r_{zW} = J_{xW zW} \qquad \sum m\left(r_{xW}^2 + r_{zW}^2\right) = I_{yW yW}$$

$$\sum m\, r_{yW}\, r_{zW} = J_{yW zW} \qquad \sum m\left(r_{xW}^2 + r_{yW}^2\right) = I_{zW zW}$$

Therefore

$$\overline{H_W} = \left[\omega_{xW} I_{xW xW} - \omega_{yW} J_{xW yW} - \omega_{zW} J_{xW zW}\right]\overline{W_1}$$
$$+\left[\omega_{yW} I_{yW yW} - \omega_{xW} J_{xW yW} - \omega_{zW} J_{yW zW}\right]\overline{W_2}$$
$$+\left[\omega_{zW} I_{zW zW} - \omega_{xW} J_{xW zW} - \omega_{yW} J_{yW zW}\right]\overline{W_3}$$

Letting

$$h_{xw} = [\omega_{xw} I_{xwxw} - \omega_{yw} J_{xwyw} - \omega_{zw} J_{xwzw}]$$

$$h_{yw} = [\omega_{yw} I_{ywyw} - \omega_{xw} J_{xwyw} - \omega_{zw} J_{ywzw}]$$

$$h_{zw} = [\omega_{zw} I_{zwzw} - \omega_{xw} J_{xwzw} - \omega_{yw} J_{ywzw}]$$

then

$$\overline{H}_W = (h_{xW})\overline{w}_1 + (h_{yW})\overline{w}_2 + (h_{zW})\overline{w}_3$$

According to Newton's Law, the vector resultant of the applied moments is equal to the time derivative of the moment of momentum ($\overline{H}$). Examination of the above equation indicates that the moment of momentum with respect to wind axes ($\overline{H}_W$) is a function of the magnitudes of the projections of the aircraft rotational velocity vector ($\overline{\omega}_A$) on the wind axes, the various products and moments of inertia of the aircraft with respect to the wind axes and the wind axes unit vectors ($\overline{w}_1$, $\overline{w}_2$, $\overline{w}_3$). The aircraft rotational velocity vector will vary with time and consequently so will the magnitudes of its projections on the wind axes. Since the wind axes origin is fixed at the aircraft center of gravity and the angles α (angle of attack) and β (yaw angle) will vary with time, the unit vectors ($\overline{w}_1$, $\overline{w}_2$, $\overline{w}_3$) must also be time dependent. Finally, the products and moments of inertia vary with the relative position of the aircraft with respect to the wind axes system. The relative position of the aircraft with respect to the wind axes is determined by the angles α and β .

Since α and β are time dependent, the products and moments of inertia must also be functions of time. In short, all the terms which constitute ($\bar{H}_W$) are time dependent. This must be taken into consideration when differentiating ($\bar{H}_W$) with respect to time.

Taking the time derivative of $(\bar{H}_W)$

$$\bar{M} = \frac{d}{dt}\left[\bar{H}_W\right] = \left\{\left[\frac{d(h_{xw})}{dt}\right]\bar{w}_1 + \left[\frac{d(h_{yw})}{dt}\right]\bar{w}_2 + \left[\frac{d(h_{zw})}{dt}\right]\bar{w}_3 + (h_{xw})\frac{d(\bar{w}_1)}{dt} + (h_{yw})\frac{d(\bar{w}_2)}{dt} + (h_{zw})\frac{d(\bar{w}_3)}{dt}\right\}$$

WHERE

$$\frac{d(h_{xw})}{dt} = \left[\dot{\omega}_{xw} I_{xwxw} - \dot{\omega}_{yw} J_{xwyw} - \dot{\omega}_{zw} J_{xwzw} + \omega_{xw}\dot{I}_{xwxw} - \omega_{yw}\dot{J}_{xwyw} - \omega_{zw}\dot{J}_{xwzw}\right]$$

$$\frac{d(h_{yw})}{dt} = \left[\dot{\omega}_{yw} I_{ywyw} - \dot{\omega}_{xw} J_{xwyw} - \dot{\omega}_{zw} J_{ywzw} + \omega_{yw}\dot{I}_{ywyw} - \omega_{xw}\dot{J}_{xwyw} - \omega_{zw}\dot{J}_{ywzw}\right]$$

$$\frac{d(h_{zw})}{dt} = \left[\dot{\omega}_{zw} I_{zwzw} - \dot{\omega}_{xw} J_{xwzw} - \dot{\omega}_{yw} J_{ywzw} + \dot{\omega}_{zw}\dot{I}_{zwzw} - \omega_{xw}\dot{J}_{xwzw} - \omega_{yw}\dot{J}_{ywzw}\right]$$

$$\frac{d(\bar{w}_1)}{dt} = \bar{\omega}_W \times \bar{w}_1 = r_W\bar{w}_2 - q_W\bar{w}_3$$

$$\frac{d(\bar{w}_2)}{dt} = \bar{\omega}_W \times \bar{w}_2 = p_W\bar{w}_3 - r_W\bar{w}_1$$

$$\frac{d(\bar{w}_3)}{dt} = \bar{\omega}_W \times \bar{w}_3 = q_W\bar{w}_1 - p_W\bar{w}_2$$

Substituting the above six derivatives and the expressions for (h_{x_W}), (h_{y_W}), AND (h_{z_W}) into the $(\overline{M})$ equation, letting $[\overline{M} = M_{x_W}\overline{w_1} + M_{y_W}\overline{w_2} + M_{z_W}\overline{w_3}]$ and equating components gives the following "moment equations of motion" with respect to the wind axes;

(M_{x_W}), (M_{y_W}), (M_{z_W}) being the <u>magnitudes</u> of the projections on the respective (X_W), (Y_W), (Z_W) wind axes of the vector resultant of the external moments applied to the aircraft.

Moment Equations with respect to Wind Axes

$$M_{x_W} = \dot{\omega}_{x_W} I_{x_W x_W} - \omega_{y_W} r_W I_{y_W y_W} + \omega_{z_W} q_W I_{z_W z_W} + (\omega_{x_W} r_W - \dot{\omega}_{y_W}) J_{x_W y_W} - (\omega_{x_W} q_W + \dot{\omega}_{z_W}) J_{x_W z_W} + (\omega_{z_W} r_W - \omega_{y_W} q_W) J_{y_W z_W} + \omega_{x_W} \dot{I}_{x_W x_W} - \omega_{y_W} \dot{J}_{x_W y_W} - \omega_{z_W} \dot{J}_{x_W z_W}$$

$$M_{y_W} = \dot{\omega}_{y_W} I_{y_W y_W} - \omega_{z_W} p_W I_{z_W z_W} + \omega_{x_W} r_W I_{x_W x_W} + (\omega_{y_W} p_W - \dot{\omega}_{z_W}) J_{y_W z_W} - (\omega_{y_W} r_W + \dot{\omega}_{x_W}) J_{x_W y_W} + (\omega_{x_W} p_W - \omega_{z_W} r_W) J_{x_W z_W} + \omega_{y_W} \dot{I}_{y_W y_W} - \omega_{z_W} \dot{J}_{y_W z_W} - \omega_{x_W} \dot{J}_{x_W y_W}$$

$$M_{z_W} = \dot{\omega}_{z_W} I_{z_W z_W} - \omega_{x_W} q_W I_{x_W x_W} + \omega_{y_W} p_W I_{y_W y_W} + (\omega_{z_W} q_W - \dot{\omega}_{x_W}) J_{x_W z_W} - (\omega_{z_W} p_W + \dot{\omega}_{y_W}) J_{y_W z_W} + (\omega_{y_W} q_W - \omega_{x_W} p_W) J_{x_W y_W} + \omega_{z_W} \dot{I}_{z_W z_W} - \omega_{x_W} \dot{J}_{x_W z_W} - \omega_{y_W} \dot{J}_{y_W z_W}$$

WHERE

$$\omega_{x_W} = [p_A \cos\alpha \cos\beta + q_A \sin\beta + r_A \sin\alpha \cos\beta]$$

$$\omega_{y_W} = [-p_A \cos\alpha \sin\beta + q_A \cos\beta - r_A \sin\alpha \sin\beta]$$

$$\omega_{z_W} = [-p_A \sin\alpha + r_A \cos\alpha]$$

The following equations are repeated from page 6

force equations with respect to wind axes

$$E_{xw} = M\frac{dV}{dt}$$

$$E_{yw} = MVr_w$$

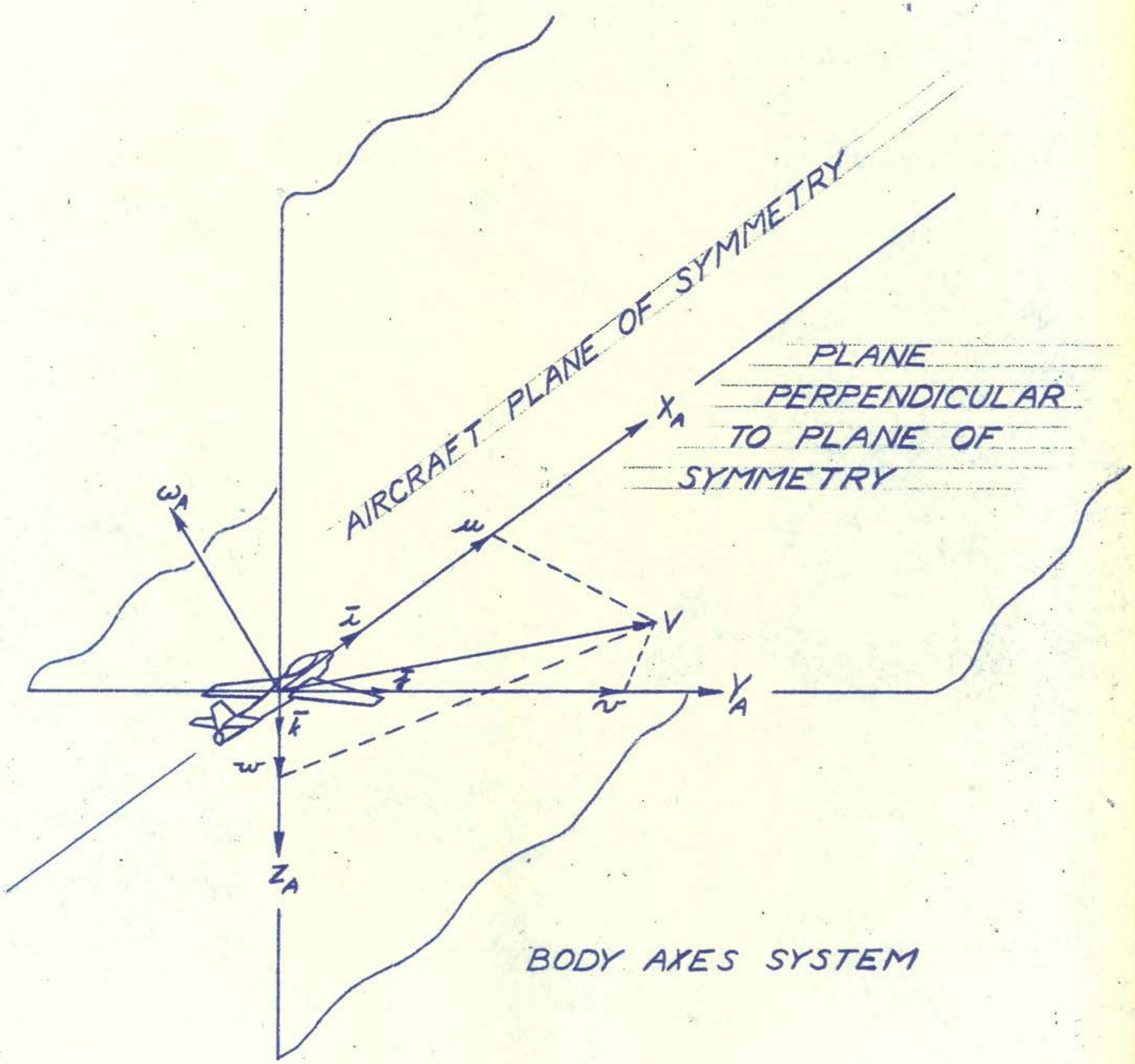
$$E_{zw} = -MVq_w$$

The above six equations constitute the general equations of motion of the aircraft as derived with respect to the wind axes reference system. The only condition imposed is the assumption of constant mass.

APPENDIX 2

Derivation of Aircraft Equations of Motion

with respect to Body Axes



BODY AXES SYSTEM

ω_A = Magnitude of the aircraft rotation vector

V = Magnitude of the aircraft velocity vector

(u, v, w) = Magnitudes of the projections of the aircraft velocity vector on the (X_A, Y_A, Z_A) body axes.

Origin of the body axes is at aircraft center of gravity

X_A = Body X axis; fixed in aircraft in plane of symmetry

Y_A = Body Y axis; perpendicular to plane of symmetry

Z_A = Body Z axis; in plane of symmetry perpendicular to X and Y body axes.

$\bar{i}$ = Unit vector in X_A direction

$\bar{j}$ = Unit vector in Y_A direction

$\bar{k}$ = Unit vector in Z_A direction

Note: The position of the aircraft relative to the body axes does not change.

I. Introduction

The equations of motion will be evolved by the application of Newton's Laws

$$\bar{F} = \frac{d}{dt}\left[\bar{Q}\right] \quad \text{AND} \quad \bar{M} = \frac{d}{dt}\left[\bar{H}\right]$$

where $\bar{Q} = \sum m\bar{V}$ is the linear momentum of the aircraft

$\bar{H} = \sum m(\bar{r}\times\bar{V})$ is the moment of momentum of the aircraft

$\bar{V}$ = the total velocity vector of a generic mass particle in the aircraft

m = the mass of a generic mass particle in the aircraft

$\bar{r}$ = the position vector of the generic mass particle " m "; measured with respect to the body axes system.

II. Derivation of force equations with respect to body axes

For the generic particle " m ", the total velocity vector is

$$\bar{V} = \hat{V} + (\bar{\omega}_A \times \bar{r})$$

where $\hat{V} = \left[u\bar{i} + v\bar{j} + w\bar{k}\right]$ is the velocity vector of the aircraft center of gravity.

The magnitude of $\hat{V}$ is

$$|\hat{V}| = V = \sqrt{u^2 + v^2 + w^2}$$

(u, v, w) are the magnitudes of the projection of $\hat{V}$ on the respective (X, Y, Z) body axes

$\bar{\omega}_A$ = the total rotational vector of the aircraft

The linear momentum of the aircraft with respect to body axes therefore becomes

$$\bar{Q} = \sum m\bar{V} = \sum m\left[\hat{V} + (\bar{\omega}_A \times \bar{r})\right]$$

$$\bar{Q} = \sum m\hat{V} + \sum m(\bar{\omega}_A \times \bar{r})$$

The velocity vector of the aircraft center of gravity ($\hat{V}$) is the same for each particle in the aircraft.
In addition, the rotation vector ($\bar{\omega}_A$) is the same for each particle in the aircraft. Therefore, in the expression for linear momentum the term ($\hat{V}$) and ($\bar{\omega}_A$) can be placed outside the summation signs. This allows ($\bar{Q}$) to be written as

$$\bar{Q} = \hat{V}\sum m + \bar{\omega}_A \times \sum m\bar{r}$$

Now ($\sum m$) is the summation of all the mass particles in the aircraft, or

$$\sum m = M \quad = \text{ total mass of aircraft}$$

($\sum m\bar{r}$) is the summation of the vector mass moments with respect to the body axes system of all the mass particles in the aircraft. Now the body axes have their origin at the aircraft center of gravity. Since the generic particle position vector is therefore indexed to the aircraft center of gravity, by virtue of the definition of center of gravity

$$\sum m\bar{r} = 0$$

Therefore

$$\bar{Q} = M\hat{V} = M(u\bar{i} + v\bar{j} + w\bar{k})$$

Taking the time derivative of the evolved expression for linear momentum gives

$$\bar{F} = \frac{d}{dt}\left[M(u\bar{i} + v\bar{j} + w\bar{k})\right]$$

<u>Assumption:</u> The mass of the aircraft is considered to be constant.

Therefore

$$\bar{F} = M\left[\left(\frac{du}{dt}\right)\bar{i} + \left(\frac{dv}{dt}\right)\bar{j} + \left(\frac{dw}{dt}\right)\bar{k}\right] + M\left[u\left(\frac{d\bar{i}}{dt}\right) + v\left(\frac{d\bar{j}}{dt}\right) + w\left(\frac{d\bar{k}}{dt}\right)\right]$$

Where the contribution $\left[\left(\frac{du}{dt}\right)\bar{i} + \left(\frac{dv}{dt}\right)\bar{j} + \left(\frac{dw}{dt}\right)\bar{k}\right]$ accounts for the rate of change with time of the magnitude (V) of the center of gravity velocity vector ($\hat{V}$).

The contribution $\left[u\left(\frac{d\bar{i}}{dt}\right) + v\left(\frac{d\bar{j}}{dt}\right) + w\left(\frac{d\bar{k}}{dt}\right)\right]$ accounts for the rotation in space of the body axes system upon which the velocity vector ($\hat{V}$) has been projected to obtain its components ($u\bar{i}$), ($v\bar{j}$) and ($w\bar{k}$).

More precisely, the term $\left[u\left(\frac{d\bar{i}}{dt}\right) + v\left(\frac{d\bar{j}}{dt}\right) + w\left(\frac{d\bar{k}}{dt}\right)\right]$ accounts for the change in direction with time of the velocity vector ($\hat{V}$). It is well to point out that (V), the magnitude of ($\hat{V}$) is the true airspeed of the aircraft.

Now $$\frac{d\bar{i}}{dt} = \bar{\omega}_A \times \bar{i} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ p_A & q_A & r_A \\ 1 & 0 & 0 \end{vmatrix} = r_A\bar{j} - q_A\bar{k}$$

$$\frac{d\bar{j}}{dt} = \bar{\omega}_A \times \bar{j} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ p_A & q_A & r_A \\ 0 & 1 & 0 \end{vmatrix} = p_A \bar{k} - r_A \bar{i}$$

$$\frac{d\bar{k}}{dt} = \bar{\omega}_A \times \bar{k} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ p_A & q_A & r_A \\ 0 & 0 & 1 \end{vmatrix} = q_A \bar{i} - p_A \bar{j}$$

where (p_A), (q_A) and (r_A) are the magnitudes of the projections of the rotation vector ($\bar{\omega}_A$) on the respective (X_A), (Y_A) and (Z_A) body axes

Substituting into the second equation on the preceding page the expressions for the time derivatives of the unit vectors and letting

$$\frac{du}{dt} = \dot{u} \ , \ \frac{dv}{dt} = \dot{v} \ ; \ \frac{dw}{dt} = \dot{w}$$

gives

$$\bar{F} = M\left[\dot{u}\bar{i} + \dot{v}\bar{j} + \dot{w}\bar{k}\right] + M\left[u(r_A\bar{j} - q_A\bar{k}) + v(p_A\bar{k} - r_A\bar{i}) + w(q_A\bar{i} - p_A\bar{j})\right]$$

$$\bar{F} = M\left[(\dot{u} - v r_A + w q_A)\bar{i} + (\dot{v} - w p_A + u r_A)\bar{j} + (\dot{w} - u q_A + v p_A)\bar{k}\right]$$

The term on the left side of the equation ($\bar{F}$), is the vector resultant of the forces applied to the aircraft. ($\bar{F}$) can be projected onto the body axes as

$$\bar{F} = E_{x_A}\bar{i} + E_{y_A}\bar{j} + E_{z_A}\bar{k}$$

For the vector equation at the bottom of the preceding page to hold, there must be an equality of components between the left and right sides of the equation. Therefore

$$E_{x_A} = M(\dot{u} - v r_A + w q_A)$$

$$E_{y_A} = M(\dot{v} - w p_A + u r_A)$$

$$E_{z_A} = M(\dot{w} - u q_A + v p_A)$$

which constitute the three "force equations" written with respect to body axes, and where (E_{x_A}), (E_{y_A}) AND (E_{z_A}) are the magnitudes of the projections on the body axes of the applied force vector ($\bar{F}$); (u), (v), (w) are the magnitudes of the projections on the body axes of the aircraft center of gravity velocity vector ($\hat{V}$); (p_A), (q_A) and (r_A) are the magnitudes of the projections on the body axes of the aircraft rotation vector ($\bar{\omega}_A$). The applied force vector ($\bar{F}$), center of gravity velocity vector ($\hat{V}$) and aircraft rotation vector ($\bar{\omega}_A$) are measured relative to fixed space references; it is their projections on the body axes that occur in the above "force equations" and in the following "moment equations".

III. Derivation of moment equations with respect to body axes.

Starting with the general expression for moment of momentum $\bar{H} = \sum m(\bar{r} \times \bar{V})$

and the total velocity vector for the generic mass particle "m"

$$\bar{V} = \hat{V} + (\bar{\omega}_A \times \bar{r})$$

the moment of momentum with respect to body axes is

$$\bar{H} = \sum m \left\{ \bar{r} \times \left[\hat{V} + (\bar{\omega}_A \times \bar{r}) \right] \right\}$$

$$\bar{H} = \sum m \left[\bar{r} \times \hat{V} \right] + \sum m \left[\bar{r} \times (\bar{\omega}_A \times \bar{r}) \right]$$

For reasons given in the derivation of the "force equations", the term

$$\sum m(\bar{r} \times \hat{V}) = -\sum m(\hat{V} \times \bar{r}) = -\hat{V} \times \sum m\bar{r} = 0$$

$$\text{SINCE } \sum m\bar{r} = 0$$

Therefore the moment of momentum becomes

$$\bar{H} = \sum m \left[\bar{r} \times (\bar{\omega}_A \times \bar{r}) \right]$$

and evaluating the double vector product $\left[\bar{r} \times (\bar{\omega}_A \times \bar{r}) \right]$ gives

$$\begin{aligned} \bar{H} = & \sum m\left[p_A(r_y^2 + r_z^2) - q_A r_x r_y - r_A r_x r_z\right]\bar{i} \\ & + \sum m\left[q_A(r_x^2 + r_z^2) - p_A r_x r_y - r_A r_y r_z\right]\bar{j} \\ & + \sum m\left[r_A(r_x^2 + r_y^2) - p_A r_x r_z - q_A r_y r_z\right]\bar{k} \end{aligned}$$

where (r_x), (r_y) and (r_z) are the magnitudes of the projections on the respective (X_A), (Y_A) and (Z_A) body axes of the generic mass particle position vector ($\bar{r}$)

As indicated in the derivation of the "force equations", the aircraft rotation vector ($\bar{\omega}_A$) can be taken outside the summation sign because it does not vary from mass particle to mass particle. The same effect applies to the magnitudes of the projections on the body axes of the rotation vector ($\bar{\omega}_A$). Therefore

$$\bar{H} = \left[p_A \sum m(r_y^2 + r_z^2) - q_A \sum m r_x r_y - r_A \sum m r_x r_z\right]\bar{i} + \left[q_A \sum m(r_x^2 + r_z^2) - p_A \sum m r_x r_y - r_A \sum m r_y r_z\right]\bar{j} + \left[r_A \sum m(r_x^2 + r_y^2) - p_A \sum m r_x r_z - q_A \sum m r_y r_z\right]\bar{k}$$

The summation quantities in the above expression define the various products and moments of inertia of the aircraft with respect to the body axes.

$$\sum m r_x r_y = J_{xy} \qquad \sum m(r_y^2 + r_z^2) = I_{xx}$$

$$\sum m r_x r_z = J_{xz} \qquad \sum m(r_x^2 + r_z^2) = I_{yy}$$

$$\sum m r_y r_z = J_{yz} \qquad \sum m(r_x^2 + r_y^2) = I_{zz}$$

Therefore

$$\bar{H} = \left[p_A I_{xx} - q_A J_{xy} - r_A J_{xz}\right]\bar{i} + \left[q_A I_{yy} - p_A J_{xy} - r_A J_{yz}\right]\bar{j} + \left[r_A I_{zz} - p_A J_{xz} - q_A J_{yz}\right]\bar{k}$$

Letting

$$h_x = \left[p_A I_{xx} - q_A J_{xy} - r_A J_{xz}\right]$$

$$h_y = \left[q_A I_{yy} - p_A J_{xy} - r_A J_{yz}\right]$$

$$h_z = \left[r_A I_{zz} - p_A J_{xz} - q_A J_{yz}\right]$$

$$\bar{H} = \left[(h_x)\bar{i} + (h_y)\bar{j} + (h_z)\bar{k}\right]$$

According to Newton's Law, the vector resultant of the applied moments is equal to the time derivative of the moment of momentum ($\bar{H}$). In the expression on the preceding page for ($\bar{H}$), the only time dependent quantities, assuming constant mass and rigid body, are the projections (p_A), (q_A) and (r_A) of the rotation vector ($\bar{\omega}_A$) and the unit direction vectors ($\bar{i}$), ($\bar{j}$) and ($\bar{k}$). Since the body axes by definition are fixed within the aircraft and no relative motion can therefore exist between the aircraft and body axes, if constant the mass and rigid body is assumed, the products and moments of inertia, which are a function of mass and position, are independent of time. This will be taken into account in obtaining the time derivative of ($\bar{H}$). The time independance of products and moments of inertia is a unique characteristic of the body axes system and leads to a set of "moment equations" much simpler than those based on other systems, i.e. the wind axes.

Taking the time derivative of the moment of momentum

$$\bar{M} = \frac{d}{dt}\left[\bar{H}\right] = \left\{\left[\frac{d(h_x)}{dt}\right]\bar{i} + \left[\frac{d(h_y)}{dt}\right]\bar{j} + \left[\frac{d(h_z)}{dt}\right]\bar{k} + h_x\left(\frac{d\bar{i}}{dt}\right) + h_y\left(\frac{d\bar{j}}{dt}\right) + h_z\left(\frac{d\bar{k}}{dt}\right)\right\}$$

where

$$\frac{d(h_x)}{dt} = \left[\dot{p}_A I_{xx} - \dot{q}_A J_{xy} - \dot{r}_A J_{xz}\right]$$

$$\frac{d(h_y)}{dt} = \left[\dot{q}_A I_{yy} - \dot{r}_A J_{yz} - \dot{p}_A J_{xy}\right]$$

$$\frac{d(h_z)}{dt} = \left[\dot{r}_A I_{zz} - \dot{p}_A J_{xz} - \dot{q}_A J_{yz}\right]$$

$$\frac{d(\bar{i})}{dt} = \left[r_A \bar{j} - q_A \bar{k}\right]$$

$$\frac{d(\bar{j})}{dt} = \left[p_A \bar{k} - r_A \bar{i}\right]$$

$$\frac{d(\bar{k})}{dt} = \left[q_A \bar{i} - p_A \bar{j}\right]$$

Substituting the above six derivatives and the expressions for (h_x), (h_y), (h_z) into the $(\bar{M})$ equation, letting $\left[\bar{M} = M_{x_A}\bar{i} + M_{y_A}\bar{j} + M_{z_A}\bar{k}\right]$ and equating components gives the following "moment equations of motion" with respect to body axes; $(M_{x_A}), (M_{y_A}), (M_{z_A})$ being the magnitudes of the projections on the $(X_A), (Y_A), (Z_A)$ body axes of the vector resultant of the external moments applied to the aircraft.

$$M_{x_A} = \left[\dot{p}_A I_{xx} + q_A r_A (I_{zz} - I_{yy}) + (r_A p_A - \dot{q}_A) J_{xy} - (p_A q_A + \dot{r}_A) J_{xz} + (r_A^2 - q_A^2) J_{yz}\right]$$

$$M_{y_A} = \left[\dot{q}_A I_{yy} + r_A p_A (I_{xx} - I_{zz}) + (p_A q_A - \dot{r}_A) J_{yz} - (q_A r_A + \dot{p}_A) J_{xy} + (p_A^2 - r_A^2) J_{xz}\right]$$

$$M_{z_A} = \left[\dot{r}_A I_{zz} + p_A q_A (I_{yy} - I_{xx}) + (q_A r_A - \dot{p}_A) J_{xz} - (r_A p_A + \dot{q}_A) J_{yz} + (q_A^2 - p_A^2) J_{xy}\right]$$

SUMMARY

Moment equations with respect to body axes:

$$M_{xA} = \Big[\dot{p}_A I_{xx} + q_A r_A (I_{ZZ} - I_{yy}) + (r_A p_A - \dot{q}_A) J_{xy} - (p_A q_A + \dot{r}_A) J_{xZ} + (r_A^2 - q_A^2) J_{yZ}\Big]$$

$$M_{yA} = \Big[\dot{q}_A I_{yy} + r_A p_A (I_{xx} - I_{ZZ}) + (p_A q_A - \dot{r}_A) J_{yZ} - (q_A r_A + \dot{p}_A) J_{xy} + (p_A^2 - r_A^2) J_{xZ}\Big]$$

$$M_{ZA} = \Big[r_A^2 I_{ZZ} + p_A q_A (I_{yy} - I_{xx}) + (q_A r_A - \dot{p}_A) J_{xZ} - (r_A p_A + \dot{q}_A) J_{yZ} + (q_A^2 - p_A^2) J_{xy}\Big]$$

Force equations with respect to body axes

$$E_{xA} = M(\dot{u} - v\, r_A + w\, q_A)$$

$$E_{yA} = M(\dot{v} - w\, p_A + u\, r_A)$$

$$E_{ZA} = M(\dot{w} - u\, q_A + v\, p_A)$$

The above six equations constitute the general equations of motion of the aircraft as derived with respect to the body axes reference system. The only condition imposed is the assumption of constant mass.

APPENDIX 3

Components of Wind Axes System Rotational Velocity Vector

Wind Axes System Rotational Velocity Vector

Let $\bar{\Omega}$ = rotational velocity vector of wind axes with respect to body axes

$\bar{\omega}_A$ = rotational velocity vector of body axes with respect to inertial axes

$\bar{\omega}_W$ = rotational velocity vector of wind axes with respect to inertial axes

Then

$$\bar{\omega}_W = \bar{\Omega} + \bar{\omega}_A \qquad (1)$$

$$\bar{\omega}_A = p_A \bar{i} + q_A \bar{j} + r_A \bar{k} \qquad (2)$$

and since transfer equations from body axes to wind axes are

$$\bar{i} = \cos\alpha \cos\beta \, \bar{w}_1 - \cos\alpha \sin\beta \, \bar{w}_2 - \sin\alpha \, \bar{w}_3 \qquad (3)$$

$$\bar{j} = \sin\beta \, \bar{w}_1 + \cos\beta \, \bar{w}_2 \qquad (4)$$

$$\bar{k} = \sin\alpha \cos\beta \, \bar{w}_1 - \sin\alpha \sin\beta \, \bar{w}_2 + \cos\alpha \, \bar{w}_3 \qquad (5)$$

$$\therefore (6) \quad \bar{\omega}_W = \bar{\Omega} + \left[p_A \cos\alpha \cos\beta + q_A \sin\beta + r_A \sin\alpha \cos\beta\right] \bar{w}_1$$
$$+ \left[-p_A \cos\alpha \sin\beta + q_A \cos\beta - r_A \sin\alpha \sin\beta\right] \bar{w}_2$$
$$+ \left[-p_A \sin\alpha + r_A \cos\alpha\right] \bar{w}_3$$

Now $\overline{\Omega}$ has been defined as the relative rotational velocity vector of the wind axes with respect to the body axes. The only degrees of freedom between wind and body axes systems are the angles α and β. Consequently the angular velocity vector of the wind axes with respect to the body axes must be the vector sum of the angular velocities of α and β.

$$\therefore \overline{\Omega} = \overline{\dot{\alpha}} + \overline{\dot{\beta}} \qquad (7)$$

Angle of attack, α, is defined as the angle between the X_A body axis and the projection on the aircraft plane of symmetry of the translation velocity vector of the aircraft center of gravity. Positive angle of attack is measured in the body axes system as a clockwise rotation about the Y_A body axis when looking in the negative direction of the Y_A body axis. Therefore, a positive rate of change of angle of attack would be represented in body axes as a vector in the negative Y_A body axis direction.

$$\therefore \overline{\dot{\alpha}} = -\dot{\alpha}\,\overline{j} \qquad (8)$$

Resorting to the transfer equations from body axes to wind axes

$$\overline{\dot{\alpha}} = -\dot{\alpha}\sin\beta\,\overline{w_1} - \dot{\alpha}\cos\beta\,\overline{w_2} \qquad (9)$$

Side slip angle, β, is defined as the angle between the aircraft plane of symmetry and the translational velocity vector of the aircraft center of gravity. Positive side slip angle is measured in the body axes system as a clockwise rotation about a line through the aircraft center of gravity and parallel to the stability axis when looking in the positive direction of the stability axis. The positive sensing of β as defined above is for measurement of β with respect to body axes. The occurrence in the definition of the Z_S stability axis is only to peg down that axis in the body axes system about which the angular rotation occurs. Anyhow from the definition, positive $\overline{\dot{\beta}}$ is a vector in the body axes system parallel to the Z_S stability axis and in the same direction as the Z_S stability axis. Even though $\overline{\dot{\beta}}$ is a vector in the body axes system we can project it where we will, and from the foregoing we can very well project it on the stability axes system where it very obligingly shows up as simply

$$\overline{\dot{\beta}} = \dot{\beta}\,\overline{S_3} \qquad (10)$$

Transfer equations from stability axes to wind axes are

$$\bar{s}_1 = \cos\beta\,\bar{w}_1 - \sin\beta\,\bar{w}_2 \qquad (11)$$

$$\bar{s}_2 = \sin\beta\,\bar{w}_1 + \cos\beta\,\bar{w}_2 \qquad (12)$$

$$\bar{s}_3 = \bar{w}_3 \qquad (13)$$

$$\therefore\ \bar{\dot\beta} = \dot\beta\,\bar{w}_3 \qquad (14)$$

And combining equations (7), (9) and (14)

$$\bar{\Omega} = -\dot\alpha\sin\beta\,\bar{w}_1 - \dot\alpha\cos\beta\,\bar{w}_2 + \dot\beta\,\bar{w}_3 \qquad (15)$$

Combining equations (6) and (15)

$$(16)\quad \bar{\omega}_W = \left[-\dot\alpha\sin\beta + p_A\cos\alpha\cos\beta + q_A\sin\beta + r_A\sin\alpha\cos\beta\right]\bar{w}_1 + \left[-\dot\alpha\cos\beta - p_A\cos\alpha\sin\beta + q_A\cos\beta - r_A\sin\alpha\sin\beta\right]\bar{w}_2 + \left[+\dot\beta - p_A\sin\alpha + r_A\cos\alpha\right]\bar{w}_3$$

In abbreviated component form

$$\bar{\omega}_W = p_W\,\bar{w}_1 + q_W\,\bar{w}_2 + r_W\,\bar{w}_3 \qquad (17)$$

Therefore, by equality of components

$$p_W = \left[-\dot\alpha\sin\beta + p_A\cos\alpha\cos\beta + q_A\sin\beta + r_A\sin\alpha\cos\beta\right] \qquad (18)$$

$$q_W = \left[-\dot\alpha\cos\beta - p_A\cos\alpha\sin\beta + q_A\cos\beta - r_A\sin\alpha\sin\beta\right] \qquad (19)$$

$$r_W = \left[\dot\beta - p_A\sin\alpha + r_A\cos\alpha\right] \qquad (20)$$

APPENDIX 4

Equations of Transfer from Body Axes System

to Inertial Axes System and Vice Versa

Body Axes Angular Position
With Respect to Inertial Space

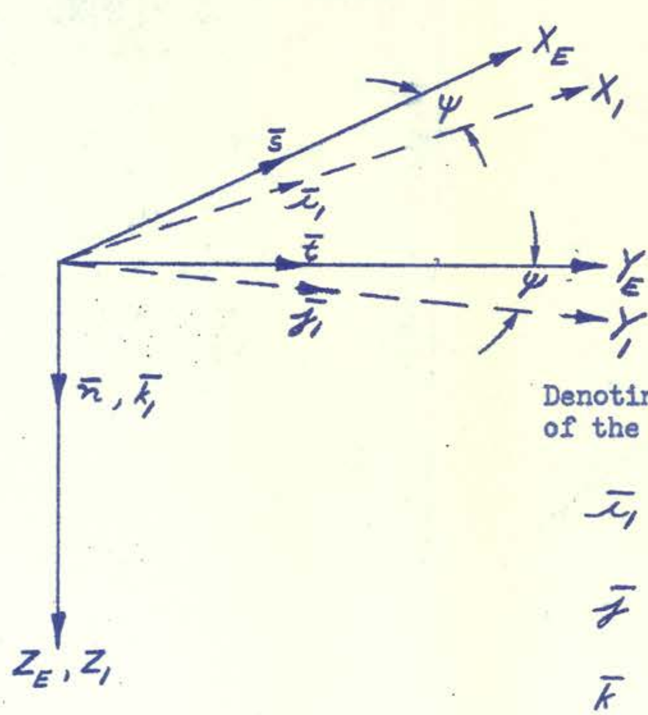
1. NOTATION

System	Axes	Unit Vectors
Inertial	X_E, Y_E, Z_E	$\bar{s}, \bar{t}, \bar{n}$
Body	X, Y, Z	$\bar{i}, \bar{j}, \bar{k}$

2. APPROACH - Angular rotation of BODY AXES with respect to INERTIAL system will follow a discreet order. We start with the BODY AXES system aligned parallel with the INERTIAL system then we rotate the BODY AXES as follows:

First, rotate the BODY AXES system an angle ψ about the BODY Z axis. Then, maintaining the BODY system in its displaced position we
Second, rotate the BODY AXES system an angle θ about the BODY Y axis. Then, maintaining the BODY system in its new displaced position we
Third, rotate the BODY AXES system an angle ϕ about the BODY X axis to obtain the final relative displacement of the BODY AXES system with respect to the INERTIAL system.

3. FIRST ROTATION ψ (HEADING)



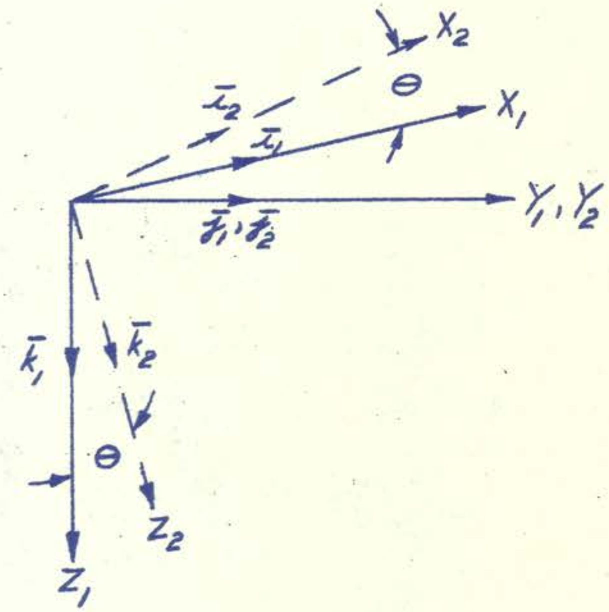
Denoting the initially displaced position of the BODY AXES with the subscript "1".

$$\bar{i}_1 = [\cos\psi]\bar{s} + [\sin\psi]\bar{t}$$

$$\bar{j} = [-\sin\psi]\bar{s} + [\cos\psi]\bar{t}$$

$$\bar{k} = \bar{n}$$

4. SECOND ROTATION Θ (PITCH)



Denoting the second displaced position of the BODY AXES by the subscript "2"

$$\bar{i}_2 = [\cos\Theta]\bar{i}_1 + [-\sin\Theta]\bar{k}_1$$

$$\bar{j}_2 = \bar{j}_1$$

$$\bar{k}_2 = [\sin\Theta]\bar{i}_1 + [\cos\Theta]\bar{k}_1$$

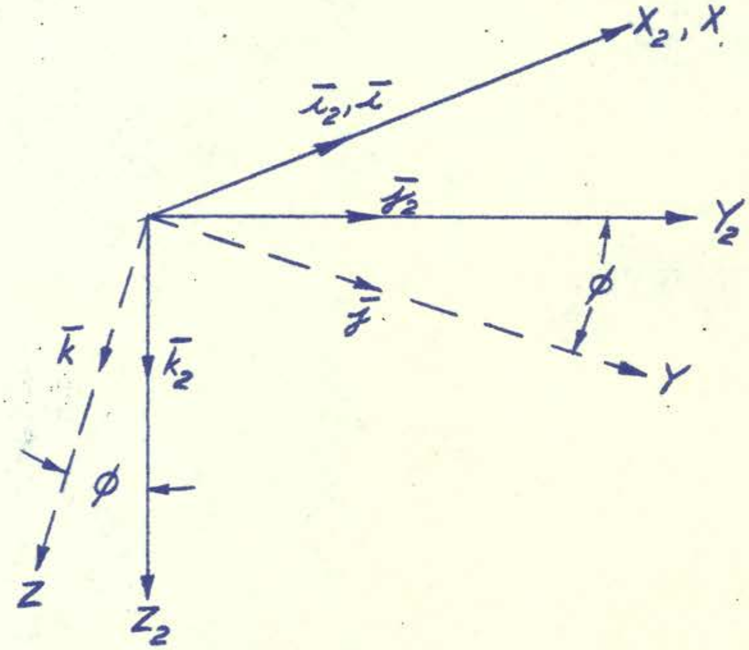
Substituting from the preceding page the expressions for $\bar{i}_1, \bar{j}_1, \text{\&} \ \bar{k}_1$ into the above equations for $\bar{i}_2, \bar{j}_2, \text{\&} \ \bar{k}_2$ gives

$$\bar{i}_2 = [\cos\Theta\cos\Psi]\bar{s} + [\cos\Theta\sin\Psi]\bar{t} - [\sin\Theta]\bar{n}$$

$$\bar{j}_2 = [-\sin\Psi]\bar{s} + [\cos\Psi]\bar{t}$$

$$\bar{k}_2 = [\sin\Theta\cos\Psi]\bar{s} + [\sin\Theta\sin\Psi]\bar{t} + [\cos\Theta]\bar{n}$$

5. THIRD AND FINAL ROTATION ϕ (ROLL)



Denoting the final position of the Body Axes by the absence of subscripts,

$$\bar{i} = \bar{i}_2$$

$$\bar{j} = [\cos\phi]\bar{j}_2 + [\sin\phi]\bar{k}_2$$

$$\bar{k} = [-\sin\phi]\bar{j}_2 + [\cos\phi]\bar{k}_2$$

Substituting from the previous page the expressions for

gives the three angular position equations which define the angular orientation of the Body Axes system with respect to the Inertial Reference System. These final equations are presented on the following page.

TRANSFER FROM BODY TO INERTIAL AXES

$$\bar{i} = \{[\cos\Theta \cos\Psi]\bar{s} + [\cos\Theta \sin\Psi]\bar{t} - [\sin\Theta]\bar{n}\}$$

$$\bar{j} = \{[\sin\phi \sin\Theta \cos\Psi - \cos\phi \sin\Psi]\bar{s} + [\sin\phi \sin\Theta \sin\Psi + \cos\phi \cos\Psi]\bar{t} + [\sin\phi \cos\Theta]\bar{n}\}$$

$$\bar{k} = \{[\cos\phi \sin\Theta \cos\Psi + \sin\phi \sin\Psi]\bar{s} + [\cos\phi \sin\Theta \sin\Psi - \sin\phi \cos\Psi]\bar{t} + [\cos\phi \cos\Theta]\bar{n}\}$$

These three simultaneous equations can be solved for $\bar{s}$, $\bar{t}$ and $\bar{n}$ to give the transfer from inertial to body system.

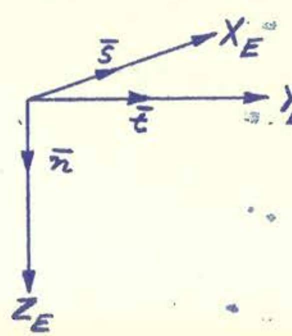
$$\bar{s} = \{[\cos\Psi \cos\Theta]\bar{i} + [\cos\Psi \sin\Theta \sin\phi - \sin\Psi \cos\phi]\bar{j} + [\cos\Psi \sin\Theta \cos\phi + \sin\Psi \sin\phi]\bar{k}\}$$

$$\bar{t} = \{[\sin\Psi \cos\Theta]\bar{i} + [\sin\Psi \sin\Theta \sin\phi + \cos\Psi \cos\phi]\bar{j} + [\sin\Psi \sin\Theta \cos\phi - \cos\Psi \sin\phi]\bar{k}\}$$

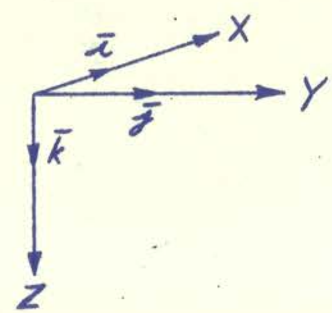
$$\bar{n} = \{[-\sin\Theta]\bar{i} + [\cos\Theta \sin\phi]\bar{j} + [\cos\Theta \cos\phi]\bar{k}\}$$

Where Ψ, Θ and ϕ are heading, pitch and roll angles of the Body Axes system as defined in text. The unit vectors $\bar{i}, \bar{j}, \bar{k}, \bar{s}, \bar{t}, \bar{n}$ are defined as follows:

INERTIAL SYSTEM



BODY AXES SYSTEM



APPENDIX 5

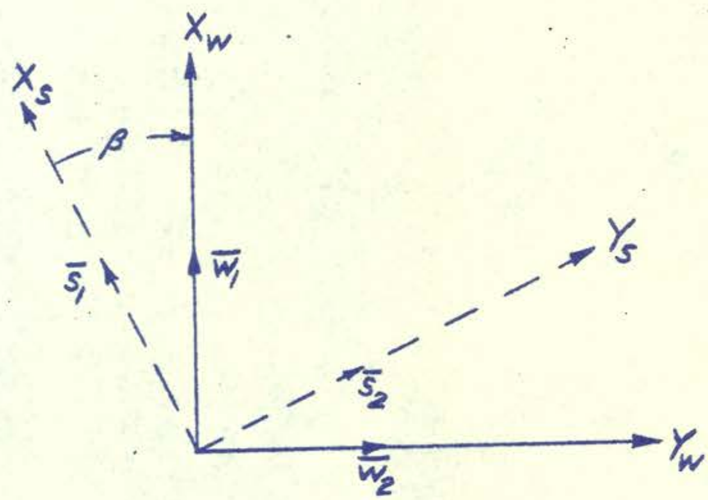
Projection of Aerodynamic Forces on Wind Axes

Let $\overline{F}$ represent the aerodynamic forces.

Then in the stability axes .

$$\overline{F} = (F_{xS})\overline{s_1} + (F_{yS})\overline{s_2} + (F_{zS})\overline{s_3}$$

Transfer equations stability to wind axes



$$\overline{s_1} = \cos\beta\,\overline{w_1} - \sin\beta\,\overline{w_2}$$

$$\overline{s_2} = \sin\beta\,\overline{w_1} + \cos\beta\,\overline{w_2}$$

$$\overline{s_3} = \overline{w_3}$$

$$\therefore\ \overline{F} = \left[F_{yS}\sin\beta + F_{xS}\cos\beta\right]\overline{w_1} + \left[F_{yS}\cos\beta - F_{xS}\sin\beta\right]\overline{w_2} + F_{zS}\,\overline{w_3}$$

Now $$F_{xS} = C_{xS}\frac{\rho V_P^2}{2}S = -C_D\frac{\rho V_P^2}{2}S$$

$$F_{yS} = C_{yS}\frac{\rho V_P^2}{2}S = C_y\frac{\rho V_P^2}{2}S$$

$$F_{zS} = C_{zS}\frac{\rho V_P^2}{2}S = -C_L\frac{\rho V_P^2}{2}S$$

$$\therefore\ \bar{F} = -\frac{\rho V_P^2 S}{2}\left[C_D\cos\beta - C_y\sin\beta\right]\bar{w}_1 + \frac{\rho V_P^2 S}{2}\left[C_D\sin\beta + C_y\cos\beta\right]\bar{w}_2 - \frac{\rho V_P^2 S}{2}\left[C_L\right]\bar{w}_3$$

where C_D, C_L, C_y are respectively the total aerodynamic drag, lift and side force coefficients as measured in the stability axes system

$$\therefore\ \bar{F} = (F_{xW})\bar{w}_1 + (F_{yW})\bar{w}_2 + (F_{zW})\bar{w}_3$$

where

$$F_{xW} = -V_P^2\frac{\rho S}{2}\left[C_D\cos\beta - C_y\sin\beta\right]$$

$$F_{yW} = V_P^2\frac{\rho S}{2}\left[C_D\sin\beta + C_y\cos\beta\right]$$

$$F_{zW} = -V_P^2\frac{\rho S}{2}\left[C_L\right]$$

APPENDIX 6

Projection of Thrust Forces on Wind Axes

In body axes thrust is

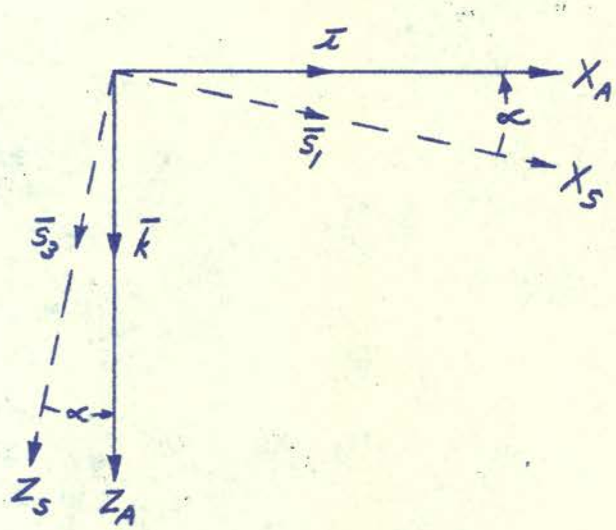
$$\bar{T} = (T_{x_A})\bar{i} + (T_{y_A})\bar{j} + (T_{z_A})\bar{k}$$

$\bar{i}$ = unit vector X_A direction

$\bar{j}$ = unit vector Y_A direction

$\bar{k}$ = unit vector Z_A direction

Transfer equations body system to stability system



$$\bar{i} = \cos\alpha\,\bar{s}_1 - \sin\alpha\,\bar{s}_3$$

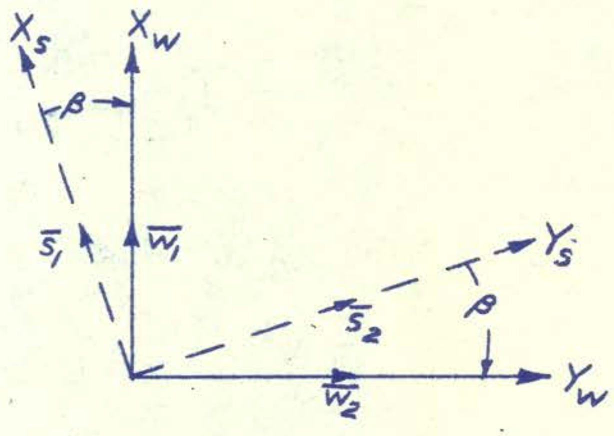
$$\bar{j} = \bar{s}_2$$

$$\bar{k} = \sin\alpha\,\bar{s}_1 + \cos\alpha\,\bar{s}_3$$

$$\therefore\ \bar{T} = (T_{x_A}\cos\alpha)\bar{s}_1 + (T_{z_A}\sin\alpha)\bar{s}_1 + (T_{y_A})\bar{s}_2 - (T_{x_A}\sin\alpha)\bar{s}_3 + (T_{z_A}\cos\alpha)\bar{s}_3$$

$$\bar{T} = (T_{x_A}\cos\alpha + T_{z_A}\sin\alpha)\bar{s}_1 + (T_{y_A})\bar{s}_2 + (-T_{x_A}\sin\alpha + T_{z_A}\cos\alpha)\bar{s}_3$$

Transfer stability to wind system



$$\bar{s}_1 = \cos\beta\ \bar{w}_1 - \sin\beta\ \bar{w}_2$$

$$\bar{s}_2 = \sin\beta\ \bar{w}_1 + \cos\beta\ \bar{w}_2$$

$$\bar{s}_3 = \bar{w}_3$$

$$\therefore\ \bar{T} = (T_{x_A}\cos\alpha\cos\beta + T_{z_A}\sin\alpha\cos\beta)\bar{w}_1 + (T_{y_A}\sin\beta)\bar{w}_1 - (T_{z_A}\sin\alpha\sin\beta + T_{x_A}\cos\alpha\sin\beta)\bar{w}_2 + (T_{y_A}\cos\beta)\bar{w}_2 + (-T_{x_A}\sin\alpha + T_{z_A}\cos\alpha)\bar{w}_3$$

$$\bar{T} = \left[T_{x_A}\cos\alpha\cos\beta + T_{y_A}\sin\beta + T_{Z_A}\sin\alpha\cos\beta\right]\bar{w}_1$$
$$\left[-T_{x_A}\cos\alpha\sin\beta + T_{y_A}\cos\beta - T_{Z_A}\sin\alpha\sin\beta\right]\bar{w}_2$$
$$\left[-T_{x_A}\sin\alpha + T_{Z_A}\cos\alpha\right]\bar{w}_3$$

$$\bar{T} = (T_{xW})\bar{w}_1 + (T_{yW})\bar{w}_2 + (T_{ZW})\bar{w}_3$$

where

$$T_{xW} = \sum_{n=1}^{\neq}\left[T_{x_A}\cos\alpha\cos\beta + T_{y_A}\sin\beta + T_{Z_A}\sin\alpha\cos\beta\right]_n$$

$$T_{yW} = \sum_{n=1}^{\neq}\left[-T_{x_A}\cos\alpha\sin\beta + T_{y_A}\cos\beta - T_{Z_A}\sin\alpha\sin\beta\right]_n$$

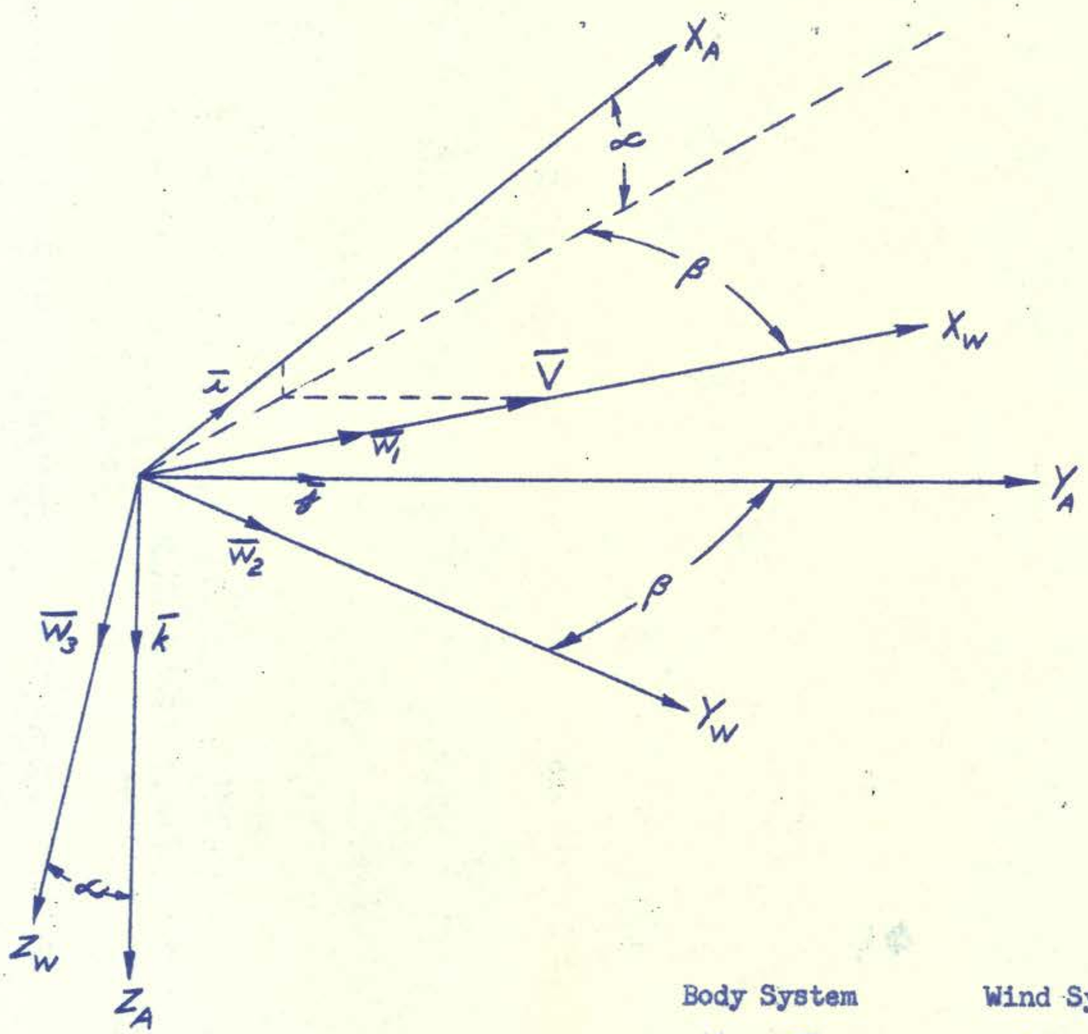
$$T_{ZW} = \sum_{n=1}^{\neq}\left[-T_{x_A}\sin\alpha + T_{Z_A}\cos\alpha\right]_n$$

$\neq$ = number of engines

APPENDIX 7

Projection of Weight on Wind Axes System

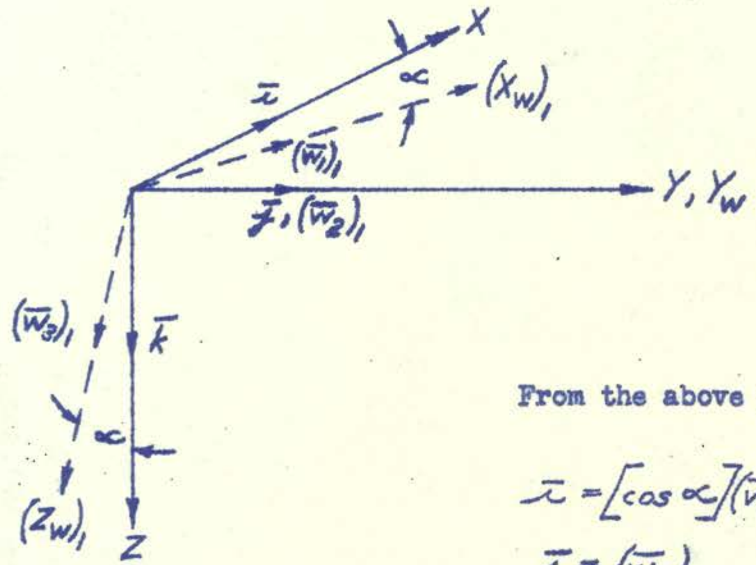
1. Relationship of Wind Axes to Body Axes



Body System		Wind System	
X_A	$\bar{i}$	X_W	$\overline{W_1}$
Y_A	$\bar{j}$	Y_W	$\overline{W_2}$
Z_A	$\bar{k}$	Z_W	$\overline{W_3}$

The Wind Axes are derived by first rotating an angle α about the Body Y Axis and then an angle β about the displaced Body Z Axis, which is the wind z axis.

2. Initial Rotation α (Angle of Attack)



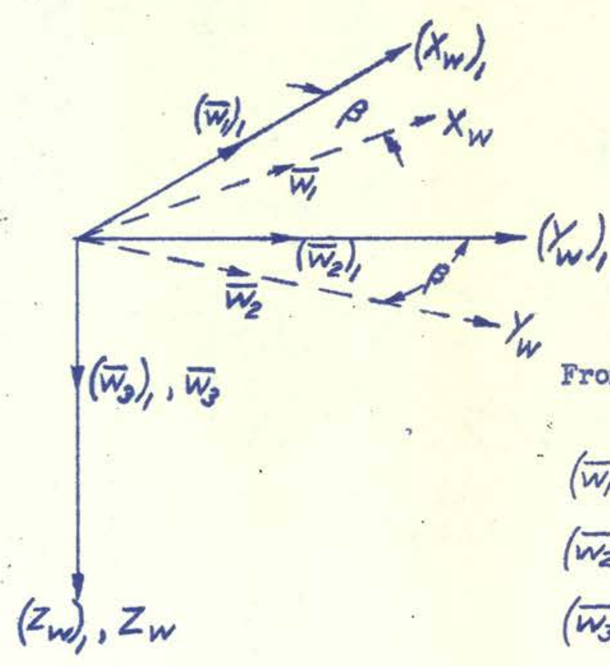
From the above sketch

$$\bar{i} = [\cos\alpha](\bar{w}_1)_1 - [\sin\alpha](\bar{w}_3)_1$$

$$\bar{j} = (\bar{w}_2)_1$$

$$\bar{k} = [\sin\alpha](\bar{w}_1)_1 + [\cos\alpha](\bar{w}_3)_1$$

3. Second and Final Rotation β (Sideslip Angle)



From the sketch to left

$$(\bar{w}_1)_1 = [\cos\beta]\bar{w}_1 - [\sin\beta]\bar{w}_2$$

$$(\bar{w}_2)_1 = [\sin\beta]\bar{w}_1 + [\cos\beta]\bar{w}_2$$

$$(\bar{w}_3)_1 = \bar{w}_3$$

4. Equations for Transfer from Body to Wind Axes

By manipulating the two sets of three equations in paragraphs 2 and 3

$$\bar{i} = \left\{ [\cos\alpha \cos\beta]\bar{w}_1 - [\cos\alpha \sin\beta]\bar{w}_2 - [\sin\alpha]\bar{w}_3 \right\}$$

$$\bar{j} = \left\{ [\sin\beta]\bar{w}_1 + [\cos\beta]\bar{w}_2 \right\}$$

$$\bar{k} = \left\{ [\sin\alpha \cos\beta]\bar{w}_1 - [\sin\alpha \sin\beta]\bar{w}_2 + [\cos\alpha]\bar{w}_3 \right\}$$

5. Projection of Weight on Inertial System

$$\bar{w} = w\,\bar{n}$$

6. Projection of weight on Body Axes - Using equations of transfer from Inertial to Body system as given in Appendix 4:

$$\bar{w} = w \left\{ [-\sin\theta]\bar{i} + [\cos\theta \sin\phi]\bar{j} + [\cos\theta \cos\phi]\bar{k} \right\}$$

7. Projection of Weight on Wind Axes - Using equations of transfer from Body to Wind System of paragraph 4 above and substituting them into the equation of paragraph 6:

$$\begin{aligned}\bar{w} = w \Big\{ & [\cos\theta \cos\phi \sin\alpha \cos\beta + \cos\theta \sin\phi \sin\beta - \sin\theta \cos\alpha \cos\beta]\bar{w}_1 \\ & + [\sin\theta \cos\alpha \sin\beta + \cos\theta \sin\phi \cos\beta - \cos\theta \cos\phi \sin\alpha \sin\beta]\bar{w}_2 \\ & + [\cos\theta \cos\phi \cos\alpha + \sin\theta \sin\alpha]\bar{w}_3 \Big\}\end{aligned}$$

APPENDIX 8

Projection Ground Reaction Forces on Wind Axes

and

Ground Reaction Moments on Body Axes

Assumptions:

1. Airframe is a rigid body with the exception of the landing gear along its line of action.
2. Mass of landing gear can be considered insignificant and gear approximated by spring and dashpot in parallel.
3. Radial distance from wheel axle to tire periphery is insignificant.
4. Aircraft axes translate with aircraft C.G., but do not rotate with respect to aircraft.
5. Each landing gear single wheel.

5.a. Landing gear remains in contact with ground so long as distance to ground along line of action is equal to or less than max extended length of gear.

6. Line of action of landing gear is perpendicular to X_a, Y_a plane in body axes system.
7. Landing gear wheels come up to speed instantaneously at ground contact.
8. Time rate of change of aircraft center of gravity position is insignificantly small.
9. Center of gravity remains in aircraft plane of symmetry and between projections on plane of symmetry of nose gear and main gear lines of action.

LANDING GEAR GROUND REACTION REFERENCE AXIS SYSTEMS

In general the individual landing gear ground reaction forces are assumed to have three components as follows:

$\tau_{1i} \Longrightarrow$ in ground X_e, Y_e plane and coincident with trace in ground plane of plane containing landing gear wheel. Positive when directed toward nose of aircraft.

$\tau_{2i} \Longrightarrow$ in ground X_e, Y_e plane and perpendicular to τ_{1i}. Positive when directed toward right wing.

$\tau_{3i} \Longrightarrow$ parallel to the Z_e axis. Positive when directed in the positive Z_e direction.

In the above notations "i" takes on value L, R, N to represent left, right and nose gear respectively.

Consider the case of roll and pitch angles both different from zero. By the definition of the roll and pitch Euler angles (ϕ, θ) of the body axes system in conjunction with assumption 6, the lines of action of all three landing gears are contained in planes parallel to the plane in which the roll angle, "ϕ", is measured.

Problem is essentially this: letting τ represent in general the main landing gear ground reaction force, resolve τ into three mutually perpendicular components, two of these components being in X_e, Y_e plane (ground plane) and the third component being perpendicular to the X_e, Y_e plane. Now consider a plane perpendicular to the axle of the wheel of the landing gear: we will refer to this as the "wheel plane". The wheel plane of the main gear is parallel to the X_a, Z_a plane (plane of symmetry). In general, because the nose wheel is steerable, this condition does not apply to the nose gear.

Anyhow, the two ground plane components of Υ should be such that one is parallel to, and the other perpendicular to, the trace of the main gear "wheel plane" on the ground plane. This is so since Υ exists only because of contact between the wheel and ground. The only mechanisms for generation of the ground reaction forces are shearing stresses and compression stresses in the runway. The component of ground reaction perpendicular to the ground plane is due to the vertical compression load, and the two horizontal components to shearing stresses.

The "wheel plane" is parallel to the aircraft plane of symmetry. Consequently the trace of the plane of symmetry on the X_e, Y_e plane will be parallel to the wheel plane trace of the main gear. Knowledge of the trace of the plane of symmetry therefore will determine the directions of Υ in the X_e, Y_e plane.

<u>GIVEN</u>:

$\bar{j}$, unit vector in positive Y_a body axis direction.

Coincidence of origins of inertial and body systems.

ψ, θ, ϕ, Euler angles body system.

$\psi = 0$.

<u>FIND</u>:

(1) Trace line in X_e, Y_e plane of plane P, (the plane of symmetry of aircraft).

(2) Components of vector $\overline{T}$ such that two components in X_e, Y_e plane, one parallel to trace line of plane P, other perpendicular to trace line of plane P. The third component to be parallel to Z_e axis.

Part (1)

Let $\overline{A}_a = X_a\,\overline{i} + Y_a\,\overline{j} + Z_a\,\overline{k}$ be position vector in body system of general point in plane "P".

Now $\overline{j}$ is perpendicular to plane of symmetry

$\therefore \overline{A}_a \perp \overline{j}$ by definition

$\therefore$ Equation of plane is:

$$\overline{A}_a \cdot \overline{j} = 0$$

$\therefore$ Cartesian form in body axes is

$$X_a\,(0) + Y_a(1) + Z_a(0) = 0$$

$$Y_a = 0$$

which is equation in body axes for plane of symmetry. But we need equation in inertial system. Let's take simple case first and impose

$$\psi = 0$$

Transfer equations body to inertial system are:

$$\left.\begin{aligned} \overline{i} &= \cos\theta\,\overline{s} - \sin\theta\,\overline{n} \\ \overline{j} &= \sin\phi\sin\theta\,\overline{s} + \cos\phi\,\overline{t} + \sin\phi\cos\theta\,\overline{n} \\ \overline{k} &= \cos\phi\sin\theta\,\overline{s} - \sin\phi\,\overline{t} + \cos\phi\cos\theta\,\overline{n} \end{aligned}\right\}\ \psi = 0$$

$$\therefore \overline{A}_e = \left\{\begin{aligned} &X_a\left[\cos\theta\,\overline{s} - \sin\theta\,\overline{n}\right] \\ &Y_a\left[\sin\phi\sin\theta\,\overline{s} + \cos\phi\,\overline{t} + \sin\phi\cos\theta\,\overline{n}\right] \\ &Z_a\left[\cos\phi\sin\theta\,\overline{s} - \sin\phi\,\overline{t} + \cos\phi\cos\theta\,\overline{n}\right] \end{aligned}\right\}\ \psi = 0$$

$$\bar{A}_e = \begin{Bmatrix} \left[X_a \cos\theta + Y_a \sin\phi \sin\theta + Z_a \cos\phi \sin\theta\right] \bar{s} = X_e \bar{s} \\ \left[Y_a \cos\phi - Z_a \sin\phi\right] \bar{t} = Y_e \bar{t} \\ \left[-X_a \sin\theta + Y_a \sin\phi \cos\theta + Z_a \cos\phi \cos\theta\right] \bar{n} = Z_e \bar{n} \end{Bmatrix}_{\psi = 0}$$

$$\therefore \bar{A}_a \cdot \bar{j} = \bar{A}_e \cdot \bar{j} = \left[X_e \bar{s} + Y_e \bar{t} + Z_e \bar{n}\right] \cdot \left[\sin\phi \sin\theta\, \bar{s} + \cos\phi\, \bar{t} + \sin\phi \cos\theta\, \bar{n}\right] = 0$$

$$\bar{A}_a \cdot \bar{j} = X_e \sin\phi \sin\theta + Y_e \cos\phi + Z_e \sin\phi \cos\theta = 0$$

which is the expression for the aircraft plane of symmetry (X_a, Z_a) plane as measured in the INERTIAL system for the special case of the body axes Euler heading angle, $\psi = 0$, and coincident origins for the body and inertial axes systems. X_e, Y_e, Z_e in the above equation are the inertial coordinates of the general point in the body axes system,

$$\bar{A}_a = X_a \bar{i} + Y_a \bar{j} + Z_a \bar{k} \qquad Y_a = 0.$$

Now, we're interested in trace of $A_a \cdot \bar{j} = 0$ on the X_e, Y_e plane. This should be given by:

$$\bar{A}_a \cdot \bar{j} = X_e \sin\phi \sin\theta + Y_e \cos\phi = 0$$

where,

$$Z_e = \left[- X_a \sin\theta + Y_a \sin\phi \cos\theta + Z_a \cos\phi \cos\theta\right] = 0$$

Therefore, the trace of the plane of symmetry in X_e, Y_e plane is given by:

$Y_e = - X_e \operatorname{Tan}\phi \sin\theta$, $\psi = 0$ and coincident origins of body and inertial axes systems.

Let ψ_{ps} represent the angle between the trace of the plane of symmetry in the X_e, Y_e plane and the X_e axis for case $\psi = 0$.

$$\psi_{ps} = \tan^{-1}\left[- \tan\phi \sin\theta\right], \qquad \psi = 0.$$

Now we've arrived at this result by considering the special case of $\psi = 0$ and coincidence of origins of body and inertial systems. If the origins are not coincident, constants will pop up in the equation which translate but do <u>not</u> rotate the plane of symmetry trace. Consequently, the angular orientation of the plane of symmetry is not altered by non-coincidence of axes systems origins. Suppose the body axes Euler angle, ψ, is different from zero. This will just rotate the trace of the plane of symmetry by an amount ψ in addition to ψ_{ps}. Consequently, we can conclude that in all generality

$$\psi_{ps} = \tan^{-1}\left[-\tan \phi \sin \theta\right]$$

represents the <u>difference</u> between the body axes Euler heading angle ψ and the heading angle of the trace of the plane of symmetry in the X_e, Y_e plane.

Just for the fun of it, and also as verification, let's obtain the trace of the plane of symmetry by another method. Now the body axis unit vector "$\bar{j}$" (unit vector Y_a direction) is perpendicular to the plane of symmetry. Trace of the plane of symmetry is contained in the plane of symmetry and consequently must be perpendicular to "$\bar{j}$". In addition, the inertial axis unit vector "$\bar{n}$" (unit vector in Z_e direction) is perpendicular to the X_e, Y_e plane. Trace of the plane of symmetry is also contained in the X_e, Y_e plane and consequently must also be perpendicular to $\bar{n}$. In short, we have a line, the trace of the plane of symmetry, simultaneously perpendicular to the two vectors "$\bar{j}$" and "$\bar{n}$". Such a relationship is prescribed by the cross product in vector rotation. Therefore, the trace of the plane of symmetry in the X_e, Y_e plane is given by:

$$\bar{d} = \bar{j} \cdot \bar{n} = \begin{vmatrix} \bar{s} & \bar{t} & \bar{n} \\ j_{xe} & j_{ye} & j_{ze} \\ 0 & 0 & 1 \end{vmatrix}$$

where $\bar{s}$, $\bar{t}$, $\bar{n}$ unit vectors in the X_e, Y_e, Z_e directions respectively and j_{xe}, j_{ye}, j_{ze} the projections of $\bar{j}$ on the X_e, Y_e, Z_e respectively.

$$\therefore \bar{d} = j_{ye}\,\bar{s} - j_{xe}\,\bar{t}$$

From page A8-5, for the case $\Psi = 0$:

$$j_{ye} = \cos\phi \;,\; j_{xe} = \sin\phi \sin\theta$$

$$\bar{d} = \cos\phi\,\bar{s} - \sin\phi \sin\theta\,\bar{t}$$

and again,

$$\psi_{ps} = \tan^{-1}\left[-\tan\phi \sin\theta\right]$$

Consider now another axes system such that:

$X_t \Longrightarrow$ parallel to trace of plane of symmetry on X_e, Y_e plane.
In other words, X_t parallel to $\bar{d} = \bar{j} \cdot \bar{n}$

$Y_t \Longrightarrow$ perpendicular to $\bar{d} = \bar{j} \cdot \bar{n}$ and contained in X_e, Y_e and contained in X_e, Y_e plane.

$Z_t \Longrightarrow$ parallel to Z_e.

and with its origin at point of contact of main gear wheel.

As far as projections of forces are concerned, the only difference between this system, which I guess we can call the "trace axes system", and the inertial axes system for the case $\Psi = 0$ is an angular rotation

$$\psi_{ps} = \tan^{-1}\left[-\tan\phi \sin\theta\right]$$

Let $\bar{T}_1$, $\bar{T}_2$, $\bar{T}_3$ be unit vectors in the X_t, Y_t Z_t directions respectively

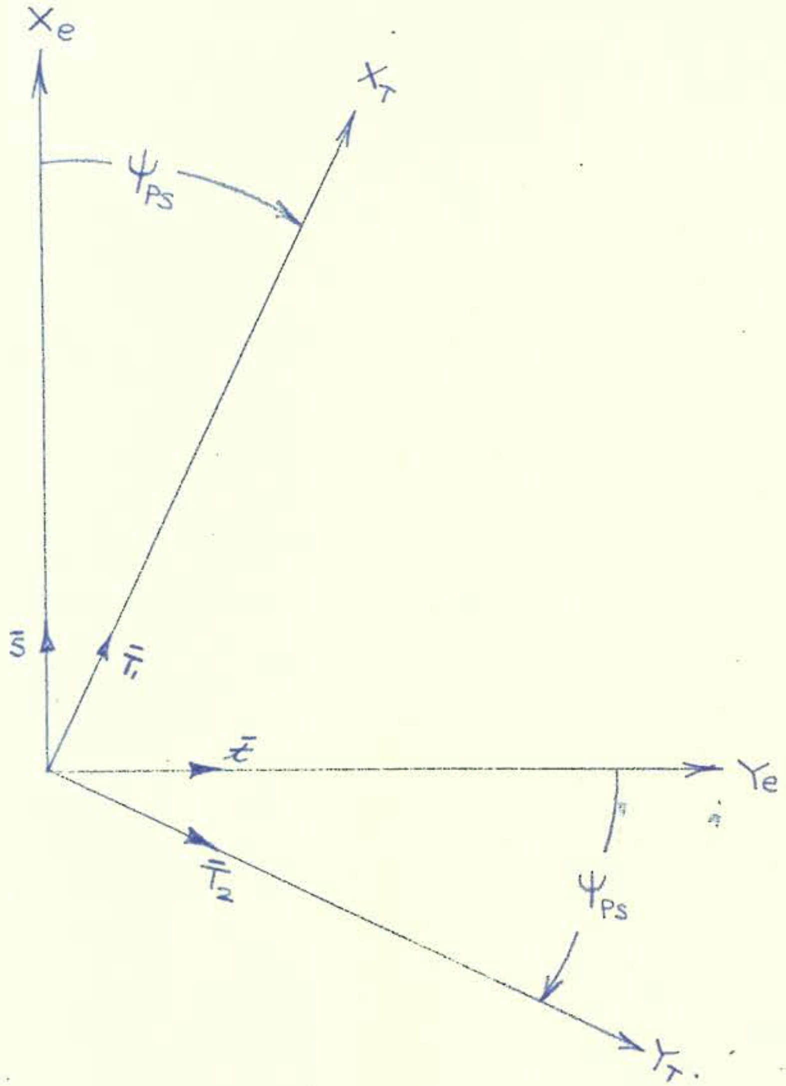
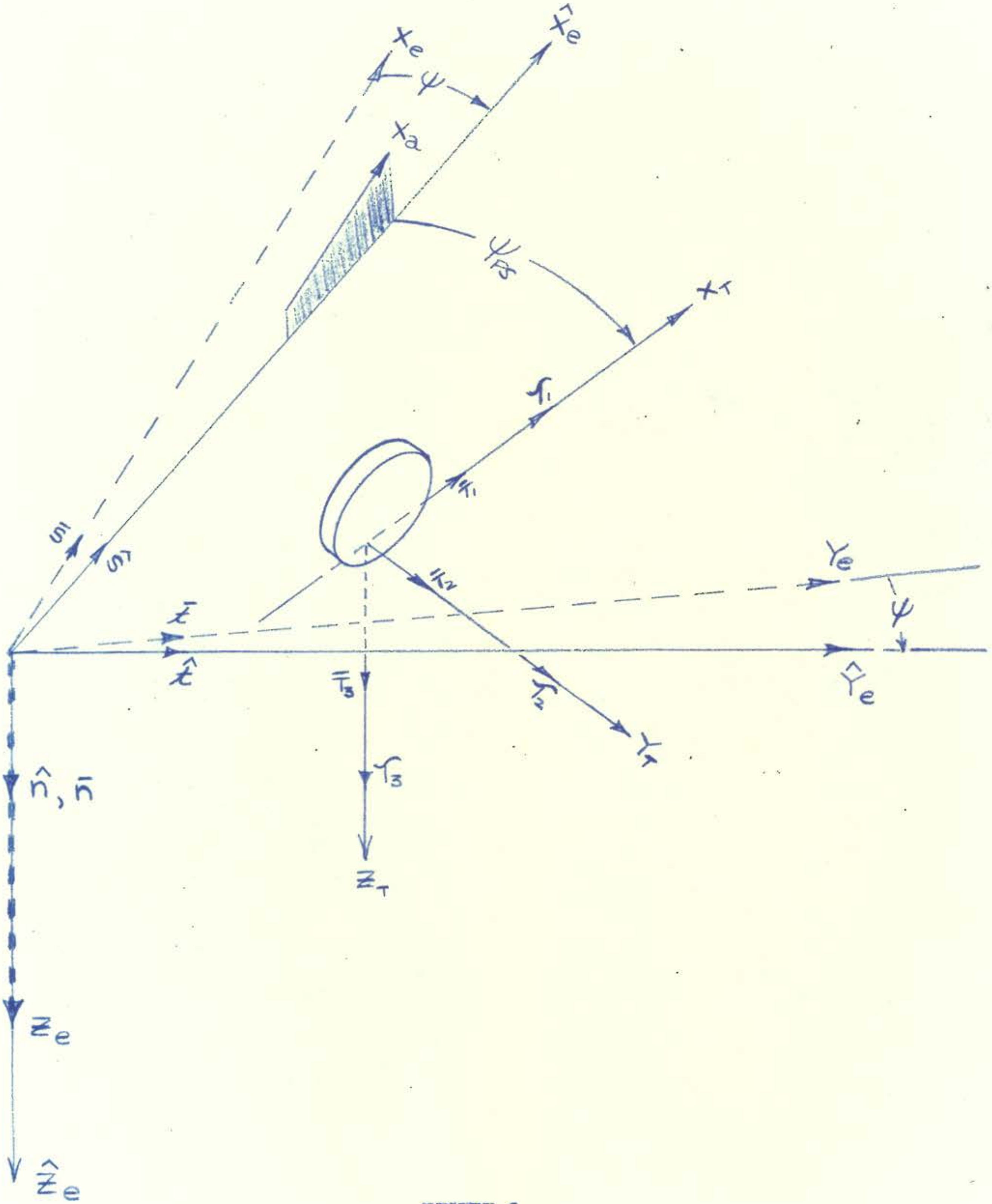


FIGURE 1

$$\therefore \bar{s} = \bar{T}_1 \cos \psi_{ps} - \bar{T}_2 \sin \psi_{ps}$$
$$\bar{t} = \bar{T}_1 \sin \psi_{ps} + \bar{T}_2 \cos \psi_{ps}$$
$$\bar{n} = \bar{T}_3$$

The significant factor is ψ_{ps}, the angle between the trace of the plane of symmetry and the projection on the X_e, Y_e plane of the X_a body axis. Therefore, let's prescribe a, so to speak, "moving inertial system", which, even though its name contains a paradox, will be useful. In essence, let's prescribe a system $\hat{X}_e$, $\hat{Y}_e$, $\hat{Z}_e$ with respective unit vectors $\hat{s}$, $\hat{t}$, $\hat{n}$ such that $\hat{s}$ is in the positive direction of the projection of X_a on the X_e, Y_e plane, $\hat{n}$ is in the direction of the Z_e axis and $\hat{t}$ completes the mutually orthogonal right handed triad. Pictorially this is shown in Figure 2. The transform equations between the body axes and the $\hat{X}_e$, $\hat{Y}_e$, $\hat{Z}_e$ system are the transform equations between the body system and inertial system for the case $\psi = 0$. And now, by gosh, we needn't care about what ψ is.



$\hat{X}_e$
X_e
ψ
X_a
ψ_β
X_T
γ_1
$\bar{\gamma}_1$
$\bar{s}$
$\hat{s}$
Y_e
$\bar{\gamma}_2$
ψ
$\bar{t}$
$\hat{t}$
$\hat{Y}_e$
$\bar{\gamma}_3$
γ_2
Y_T
$\hat{n}, \bar{n}$
γ_3
Z_T
Z_e
$\hat{Z}_e$

FIGURE 2

What we have discussed so far applies in general to the main gear, but not in general to the nosewheel. Since the nosewheel is steerable, in general the plane containing the nosewheel is not parallel to the aircraft plane of symmetry but rotated with respect to the plane of symmetry by an angle λ_N. Our problem is now to find the trace of the nosewheel plane in the X_e, Y_e plane. Okay, here goes: let $\bar{e}$ represent unit vector normal to nosewheel plane. We assume nosewheel plane rotates about the line of action of the nose gear. By assumption 6, line of action of nose gear is perpendicular to X_a, Y_a plane. Therefore, nosewheel plane is perpendicular to X_a, Y_a plane and consequently normal to nosewheel plane must be parallel to X_a, Y_a plane. From which

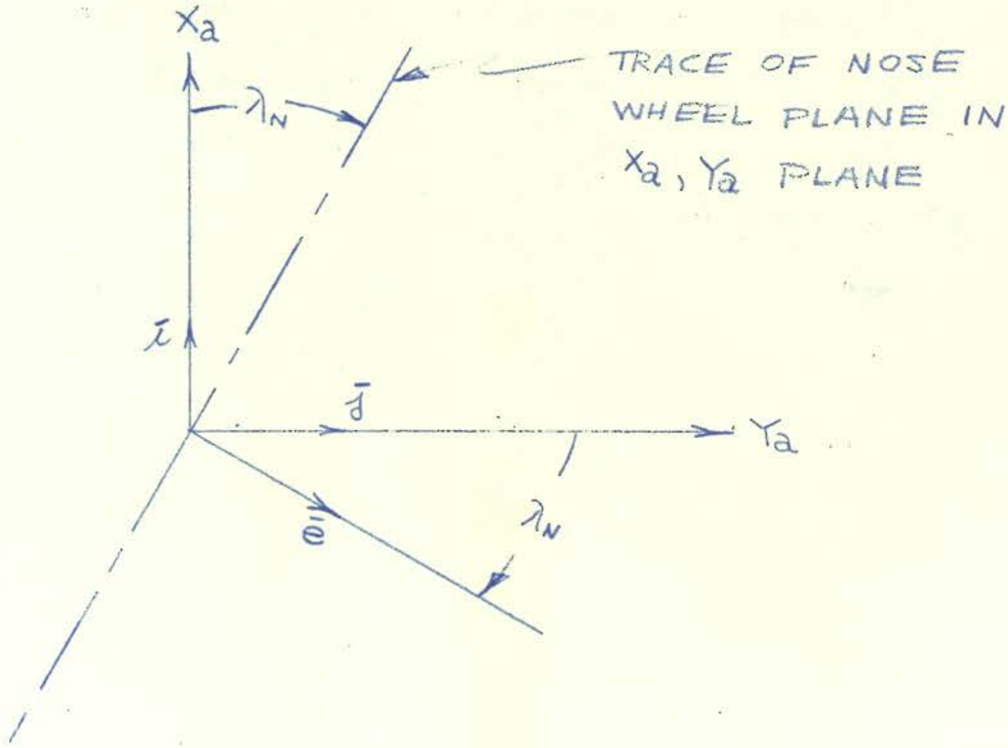


FIGURE 3

Since $\bar{e}$ is a unit vector

$$\bar{e} = -\sin\lambda_N \bar{i} + \cos\lambda_N \bar{j}$$

Now from the discussion on page A8-10, and the transform equations between body axes and inertial axes for $\psi = 0$ as given on page A8-5, the transform equations between the body axes and the X_e, Y_e, Z_e system are:

$$\bar{i} = \cos\theta\,\hat{s} - \sin\theta\,\hat{n}$$

$$\bar{j} = \sin\phi\sin\theta\,\hat{s} + \cos\phi\,\hat{t} + \sin\phi\cos\theta\,\hat{n}$$

$$\bar{k} = \cos\phi\sin\theta\,\hat{s} - \sin\phi\,\hat{t} + \cos\phi\cos\theta\,\hat{n}$$

$$\therefore e = -\sin\lambda_N\cos\theta\,\hat{s} + \sin\lambda_N\sin\theta\,\hat{n} + \cos\lambda_N\sin\phi\sin\theta\,\hat{s}$$

$$+ \cos\lambda_N\cos\phi\,\hat{t} + \cos\lambda_N\sin\phi\cos\theta\,\bar{n}$$

$$\bar{e} = \left[\cos\lambda_N\sin\phi\sin\theta - \sin\lambda_N\cos\theta\right]\hat{s}$$

$$\left[\cos\lambda_N\cos\phi\right]\hat{t}$$

$$\left[\cos\lambda_N\sin\phi\cos\theta + \sin\lambda_N\sin\theta\right]\hat{n}$$

Following approach on page A8-8 the trace of the nosewheel plane in the $\hat{X}_e$, $\hat{Y}_e$ plane is given by

$$\hat{f} = \hat{e} \times \hat{n} = \begin{vmatrix} \hat{s} & \hat{t} & \hat{n} \\ j\hat{X}_e & j\hat{Y}_e & j\hat{Z}_e \\ 0 & 0 & 1 \end{vmatrix} = j\hat{Y}_e\,\hat{s} - j\hat{X}_e\,\hat{t}$$

Obtaining $j\hat{Y}_e$ and $j\hat{X}_e$ as the $\hat{t}$ and $\hat{s}$ components respectively of $\bar{e}$ as expanded above gives:

$$\hat{f} = \left[\cos \lambda_N \cos \phi\right] \hat{s} - \left[\cos \lambda_N \sin \phi \sin \theta - \sin \lambda_N \cos \theta\right] \hat{t}$$

Now recognizing that:

$$(X_e, Y_e, Z_e) \Rightarrow (\hat{X}_e, \hat{Y}_e, \hat{Z}_e) \text{ for } \psi = 0,$$

the trace line equation for the plane of symmetry in the X_e, Y_e, Z_e system is:

$$\hat{d} = \cos \phi \, \hat{s} - \sin \phi \sin \theta \, \hat{t}.$$

Letting ψ_{NP} represent angle of nose wheel plane trace in $\hat{X}_e$, $\hat{Y}_e$ plane we get:

$$\psi_{NP} = \tan^{-1} \left[- \tan \phi \sin \theta + \frac{\tan \lambda_N \cos \theta}{\cos \phi}\right]$$

$$\psi_{PS} = \tan^{-1} \left[- \tan \phi \sin \theta\right]$$

Consider yet another axes system X_{NT}, Y_{NT}, Z_{NT} with respective unit vector $\overline{N}_1$, $\overline{N}_2$, $\overline{N}_3$ identical in concept to the X_T, Y_T, Z_T system but referenced to the trace of the nose wheel plane. Then we have as the transform equations between this new system and the $\hat{X}_e$, $\hat{Y}_e$, $\hat{Z}_e$ system.

$$\hat{s} = \overline{N}_1 \cos \psi_{NP} - \overline{N}_2 \sin \psi_{NP}$$

$$\hat{t} = \overline{N}_1 \sin \psi_{NP} + \overline{N}_2 \cos \psi_{NP}$$

$$\hat{n} = \overline{N}_3$$

The overall relationships amoung the inertial system, the $\hat{X}_e$, $\hat{Y}_e$, $\hat{Z}_e$ system, the X_T, Y_T, Z_T system and the X_{NT}, Y_{NT}, Z_{NT} system are shown in Figure 4.

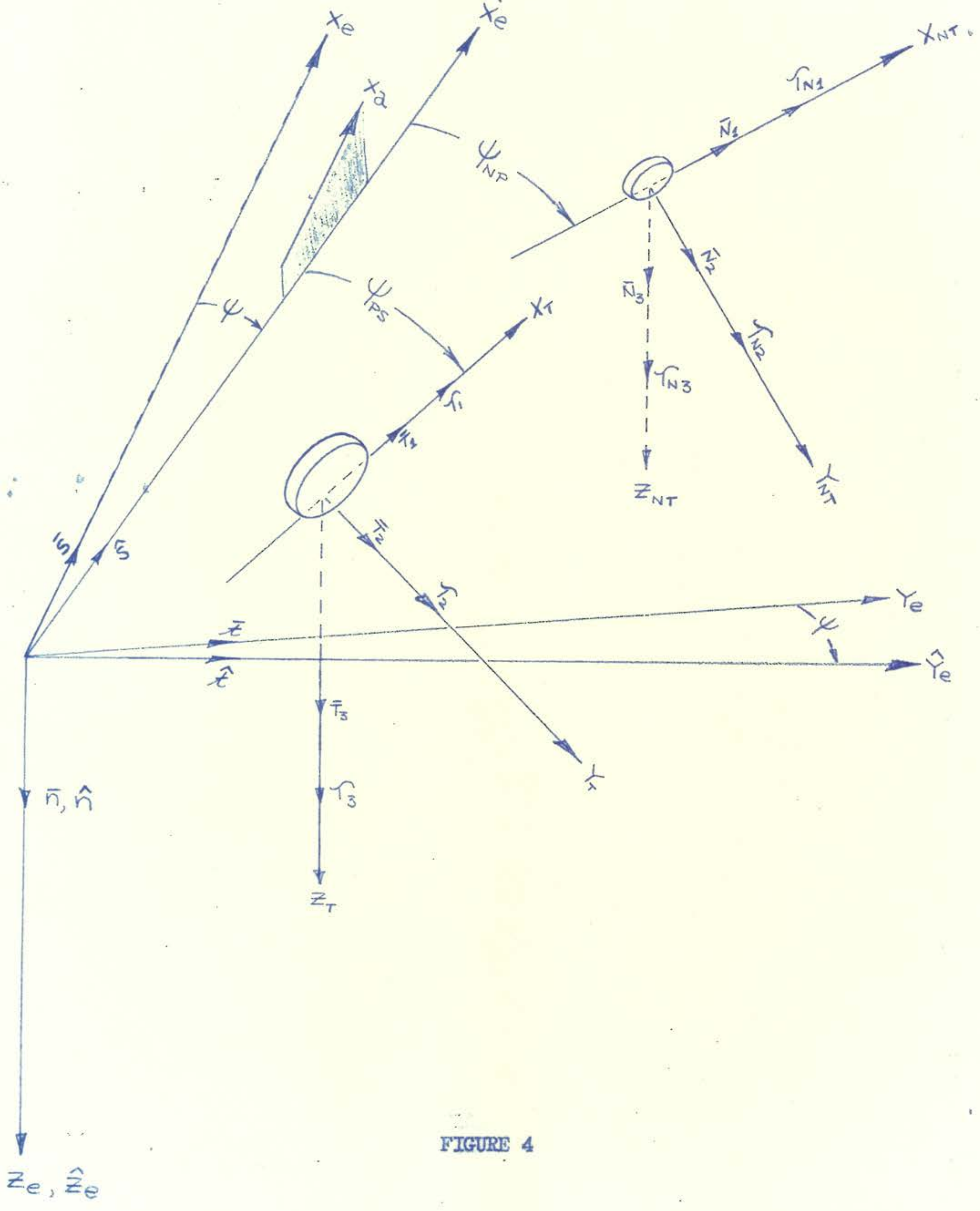

X_e
$\hat{X}_e$
X_a
X_{NT}
Υ_{N1}
$\bar{N}_1$
ψ_{NP}
$\bar{N}_2$
$\bar{N}_3$
ψ
ψ_{PS}
X_T
Υ_{N2}
Υ_{N3}
Υ_1
$\bar{T}_1$
Z_{NT}
Y_{NT}
$\bar{\xi}$
$\hat{\xi}$
$\bar{T}_2$
Υ_2
Y_e
$\bar{t}$
ψ
$\hat{t}$
$\hat{Y}_e$
$\bar{T}_3$
Y_T
$\bar{n}, \hat{n}$
Υ_3
Z_T
$Z_e, \hat{Z}_e$

FIGURE 4

PROJECTION OF GROUND REACTION FORCES ON WIND AXES

Let main gear forces be given by:

$$\overline{\tau}_M = (\tau_1)\,\overline{T}_1 + (\tau_2)\,\overline{T}_2 + (\tau_3)\,\overline{T}_3$$

where:

$$\overline{\tau}_M = \overline{\tau}_L + \overline{\tau}_R$$

$$\tau_1 = \tau_{1L} + \tau_{1R}$$

$$\tau_2 = \tau_{2L} + \tau_{2R}$$

$$\tau_3 = \tau_{3L} + \tau_{3R}$$

Transform equations to go from X_T, Y_T, Z_T system to $\hat{X}_e$, $\hat{Y}_e$, $\hat{Z}_e$ system can be evolved from Figure 2 to be:

$$\overline{T}_1 = \hat{s}\cos\psi_{PS} + \hat{t}\sin\psi_{PS}$$

$$\overline{T}_2 = -\hat{s}\sin\psi_{PS} + \hat{t}\cos\psi_{PS}$$

$$\overline{T}_3 = \hat{n}$$

$$\overline{\tau}_M = \begin{cases} (\tau_1 \cos\psi_{PS})\,\hat{s} + (\tau_1 \sin\psi_{PS})\,\hat{t} \\ -(\tau_2 \sin\psi_{PS})\,\hat{s} + (\tau_2 \cos\psi_{PS})\,\hat{t} \\ \tau_3 \hat{n} \end{cases}$$

$$\therefore \tau_M = \left[\tau_1 \cos\psi_{PS} - \tau_2 \sin\psi_{PS}\right]\hat{s} + \left[\tau_1 \sin\psi_{PS} + \tau_2 \cos\psi_{PS}\right]\hat{t} + \tau_3\,\hat{n}$$

Transform equations going from $\hat{X}_e$, $\hat{Y}_e$, $\hat{Z}_e$ system to body system are the same as the transform equations from inertial system to body system for special case $\psi = 0$. These are:

$$\hat{s} = [\cos\theta]\,\bar{i} + [\sin\theta\sin\phi]\,\bar{j} + [\sin\theta\cos\phi]\,\bar{k}$$
$$\hat{t} = [\cos\phi]\,\bar{j} - [\sin\phi]\,\bar{k}$$
$$\hat{n} = -[\sin\theta]\,\bar{i} + [\cos\theta\sin\phi]\,\bar{j} + [\cos\theta\cos\phi]\,\bar{k}$$

$$\begin{aligned}\bar{\tau}_M = & \left[\tau_1\cos\theta\cos\psi_{PS} - \tau_2\cos\theta\sin\psi_{PS}\right]\bar{i} \\ & + \left[\tau_1\sin\theta\sin\phi\cos\psi_{PS} - \tau_2\sin\theta\sin\phi\sin\psi_{PS}\right]\bar{j} \\ & + \left[\tau_1\sin\theta\cos\phi\cos\psi_{PS} - \tau_2\sin\theta\cos\phi\sin\psi_{PS}\right]\bar{k} \\ & + \left[\tau_1\cos\phi\sin\psi_{PS} + \tau_2\cos\phi\cos\psi_{PS}\right]\bar{j} \\ & - \left[\tau_1\sin\phi\sin\psi_{PS} + \tau_2\sin\phi\cos\psi_{PS}\right]\bar{k} \\ & - \left[\tau_3\sin\theta\right]\bar{i} \\ & + \left[\tau_3\cos\theta\sin\phi\right]\bar{j} \\ & + \left[\tau_3\cos\theta\cos\phi\right]\bar{k}\end{aligned}$$

where:

$$\tau_1 = \tau_{1L} + \tau_{1R}$$

$$\tau_2 = \tau_{2L} + \tau_{2R}$$

$$\tau_3 = \tau_{3L} + \tau_{3R}$$

Transform equations going from body to wind axes are:

$$\bar{i} = [\cos\alpha \cos\beta]\,\bar{w}_1 - [\cos\alpha \sin\beta]\,\bar{w}_2 - [\sin\alpha]\,\bar{w}_3$$

$$\bar{j} = [\sin\beta]\,\bar{w}_1 + [\cos\beta]\,\bar{w}_2$$

$$\bar{k} = [\sin\alpha \cos\beta]\,\bar{w}_1 - [\sin\alpha \sin\beta]\,\bar{w}_2 + [\cos\alpha]\,\bar{w}_3$$

$$\therefore \tau_M = [\tau_{M_{xA}} \cos\alpha \cos\beta]\,\bar{w}_1 - [\tau_{M_{xA}} \cos\alpha \sin\beta]\,\bar{w}_2 - [\tau_{M_{xA}} \sin\alpha]\,\bar{w}_3 + [\tau_{M_{yA}} \sin\beta]\,\bar{w}_1 + [\tau_{M_{yA}} \cos\beta]\,\bar{w}_2 + [\tau_{M_{zA}} \sin\alpha \cos\beta]\,\bar{w}_1 - [\tau_{M_{zA}} \sin\alpha \sin\beta]\,\bar{w}_2 + [\tau_{M_{zA}} \cos\alpha]\,\bar{w}_3$$

∴ Main gear ground reaction forces projected on wind axes

$$\bar{\tau}_M = [\tau_{M_{xA}} \cos\alpha \cos\beta + \tau_{M_{yA}} \sin\beta + \tau_{M_{zA}} \sin\alpha \cos\beta]\,\bar{w}_1$$
$$[-\tau_{M_{xA}} \cos\alpha \sin\beta + \tau_{M_{yA}} \cos\beta - \tau_{M_{zA}} \sin\alpha \sin\beta]\,\bar{w}_2$$
$$[-\tau_{M_{xA}} \sin\alpha + \tau_{M_{zA}} \cos\alpha]\,\bar{w}_3$$

Therefore, combining results of pages A8- 17 and A8- 18 as summary:

Main gear ground reaction forces projected on body axes:

$$\mathcal{T}_M = (\mathcal{T}_{Mx_A})\,\bar{i} + (\mathcal{T}_{My_A})\,\bar{j} + (\mathcal{T}_{Mz_A})\,\bar{k}$$

$$\mathcal{T}_{Mx_A} = \Big[(\mathcal{T}_{1L} + \mathcal{T}_{1R}) \cos\theta \cos\psi_{PS} - (\mathcal{T}_{2L} + \mathcal{T}_{2R}) \cos\theta \sin\psi_{PS} - (\mathcal{T}_{3L} + \mathcal{T}_{3R}) \sin\theta\Big]$$

$$\mathcal{T}_{My_A} = \Big[(\mathcal{T}_{1L} + \mathcal{T}_{1R}) (\sin\theta \sin\phi \cos\psi_{PS} + \cos\phi \sin\psi_{PS}) + (\mathcal{T}_{2L} + \mathcal{T}_{2R}) (\cos\phi \cos\psi_{PS} - \sin\theta \sin\phi \sin\psi_{PS}) + (\mathcal{T}_{3L} + \mathcal{T}_{3R}) (\cos\theta \sin\phi)\Big]$$

$$\mathcal{T}_{Mz_A} = \Big[(\mathcal{T}_{1L} + \mathcal{T}_{1R}) (\sin\theta \cos\phi \cos\psi_{PS} - \sin\phi \sin\psi_{PS}) - (\mathcal{T}_{2L} + \mathcal{T}_{2R}) (\sin\phi \cos\psi_{PS} + \sin\theta \cos\phi \sin\psi_{PS}) + (\mathcal{T}_{3L} + \mathcal{T}_{3R}) (\cos\theta \cos\phi)\Big]$$

Main gear ground reaction forces projected on wind axes:

$$\overline{\mathcal{T}}_M = (\mathcal{T}_{Mx_W})\,\bar{w}_1 + (\mathcal{T}_{My_W})\,\bar{w}_2 + (\mathcal{T}_{Mz_W})\,\bar{w}_3$$

$$\mathcal{T}_{Mxw} = \Big[\mathcal{T}_{Mx_A} \cos\alpha \cos\beta + \mathcal{T}_{My_A} \sin\beta + \mathcal{T}_{Mz_A} \sin\alpha \cos\beta\Big]$$

$$\mathcal{T}_{Myw} = \Big[-\mathcal{T}_{Mx_A} \cos\alpha \sin\beta + \mathcal{T}_{My_A} \cos\beta - \mathcal{T}_{Mz_A} \sin\alpha \sin\beta\Big]$$

$$\mathcal{T}_{Mz_W} = \Big[-\mathcal{T}_{Mx_A} \sin\alpha + \mathcal{T}_{Mz_A} \cos\alpha\Big]$$

Geometrically, only difference between projecting main gear graund reaction forces on body axes and nose gear ground recction forces on body axes is the difference in the angles ψ_{ps} and ψ_{np}. Consequently, by analogy to the projection of main gear ground reaction forces on the body axes, we can immediately write the projections of nose gear ground reaction forces.

Nose gear ground reaction forces projected on body axes:

$$\mathcal{T}_N = (\mathcal{T}_{Nx_A})\,\bar{i} + (\mathcal{T}_{Ny_A})\,\bar{j} + (\mathcal{T}_{Nz_A})\,\bar{k}$$

$$\mathcal{T}_{Nx_A} = \left[(\mathcal{T}_{1N})\cos\theta\cos\psi_{NP} - (\mathcal{T}_{2N})\cos\theta\sin\psi_{NP} - (\mathcal{T}_{3N})\sin\theta\right]$$

$$\mathcal{T}_{Ny_A} = \Big[(\mathcal{T}_{1N})(\sin\theta\sin\phi\cos\psi_{NP} + \cos\phi\sin\psi_{NP})$$

$$+(\mathcal{T}_{2N})(\cos\phi\cos\psi_{NP} - \sin\theta\sin\phi\sin\psi_{NP}) + (\mathcal{T}_{3N})(\cos\theta\sin\phi)\Big]$$

$$\mathcal{T}_{Nz_A} = \Big[(\mathcal{T}_{1N})(\sin\theta\cos\phi\cos\psi_{NP} - \sin\phi\sin\psi_{NP})$$

$$-(\mathcal{T}_{2N})(\sin\phi\cos\psi_{NP} + \sin\theta\cos\phi\sin\psi_{NP}) + (\mathcal{T}_{3N})(\cos\theta\cos\phi)\Big]$$

Nose gear ground reaction forces projected on wind axes:

$$\bar{\mathcal{T}}_N = (\mathcal{T}_{Nx_W})\,\bar{w}_1 + (\mathcal{T}_{Ny_W})\,\bar{w}_2 + (\mathcal{T}_{Nz_W})\,\bar{w}_3$$

$$\mathcal{T}_{Nx_W} = \left[\mathcal{T}_{Nx_A}\cos\alpha\cos\beta + \mathcal{T}_{Ny_A}\sin\beta + \mathcal{T}_{Nz_A}\sin\alpha\cos\beta\right]$$

$$\mathcal{T}_{Ny_W} = \left[-\mathcal{T}_{Nx_A}\cos\alpha\sin\beta + \mathcal{T}_{Ny_A}\cos\beta - \mathcal{T}_{Nz_A}\sin\alpha\sin\beta\right]$$

$$\mathcal{T}_{Nz_W} = \left[-\mathcal{T}_{Nx_A}\sin\alpha + \mathcal{T}_{Nz_A}\cos\alpha\right]$$

LANDING GEAR GROUND REACTION MOMENTS ABOUT BODY AXES

In the body axes system, point of application or right main gear force for a nose wheel type landing gear system is given by:

$$\bar{R}_R = (-X_{a_M})\,\bar{i} + (Y_{a_M})\,\bar{j} + (h_R)\,\bar{k}$$

where:

$X_{a_M} \Longrightarrow$ Absolute value X_a coordinate intersection main gear line of action with X_a, Y_a plane.

$Y_{a_M} \Longrightarrow$ Aboslute value Y_a coordinate intersection main gear line of action with X_a, Y_a plane.

$h_R \Longrightarrow$ Extension of right main gear.

The right main gear forces are given by:

$$\bar{\mathcal{T}}_R = \bar{\mathcal{T}}_M \Big|_{\substack{\mathcal{T}_{1L} = 0 \\ \mathcal{T}_{2L} = 0 \\ \mathcal{T}_{3L} = 0}} = (\mathcal{T}_{R_{X_a}})\,\bar{i} + (\mathcal{T}_{R_{Y_a}})\,\bar{j} + (\mathcal{T}_{R_{Z_a}})\,\bar{k}$$

The right main gear ground reaction moments about body axes are then given by:

$$\bar{M}_R = \bar{R}_R \times \bar{\mathcal{T}}_R = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -X_{a_M} & Y_{A_M} & h_R \\ \mathcal{T}_{R_{X_a}} & \mathcal{T}_{R_{Y_a}} & \mathcal{T}_{R_{Z_a}} \end{vmatrix}$$

$$\overline{M}_R = \overline{R}_R \times \overline{\tau}_R = \left[Y_{aM}\,\tau_{RZa} - h_R\,\tau_{RY_a}\right]\bar{i} + X_{aM}\,\tau_{RZa} + h_R\,\tau_{RX_a}\Big]\,\bar{j}$$

$$+ \left[-X_{aM}\,\tau_{RY_a} - Y_{aM}\,\tau_{RX_a}\right]\bar{k}$$

$$\overline{R}_L = (-X_{aM})\,\bar{i} + (-Y_{aM})\,\bar{j} + (h_L)\,\bar{k}$$

$$\overline{\tau}_L = \overline{\tau}_M \Big|_{\substack{\tau_{1R} = 0 \\ \tau_{2R} = 0 \\ \tau_{3R} = 0}} = (\tau_{LX_a})\,\bar{i} + (\tau_{LY_a})\,\bar{j} + (\tau_{LZ_A})\,\bar{k}$$

$$\therefore \overline{M}_L = \overline{R}_L \times \overline{\tau}_L = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -X_{aM} & -Y_{aM} & h_L \\ \tau_{LX_a} & \tau_{LY_a} & \tau_{LZ_a} \end{vmatrix}$$

$$\overline{M}_L = \left[-Y_{aM}\,\tau_{LZa} - h_L\,\tau_{LYa}\right]\bar{i} + \left[X_{aM}\,\tau_{LZa} + h_L\,\tau_{LX_a}\right]\bar{j}$$

$$+ \left[-X_{aM}\,\tau_{LY_a} + Y_{aM}\,\tau_{LX_a}\right]\bar{k}$$

And for the nose gear:

$$\overline{R}_N = (X_{aN})\,\bar{i} + (0)\,\bar{j} + (h_N)\,\bar{k}$$

$$\tau_N = (\overline{\tau}_{N\,xa})\,\bar{i} + (\overline{\tau}_{N_{ya}})\,\bar{j} + (\tau_{N\,za})\,\bar{k}$$

$$\overline{M}_n = \overline{R}_N \times \tau_N = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ X_{an} & 0 & h_N \\ \tau_{Nxa} & \tau_{Nya} & \tau_{N.za} \end{vmatrix}$$

$$\overline{M}_n = \left[-h_N\,\tau_{Nya}\right]\bar{i} + \left[-X_{aN}\,\tau_{N.za} + h_N\,\tau_{Nxa}\right]\bar{j} + \left[X_{aN}\,\tau_{Nya}\right]\bar{k}$$

SUMMARY MOMENTS LANDING GEAR GROUND REACTION FORCES ABOUT BODY AXES:

$$\overline{M}_G = (M_{G_{xa}})\,\overline{i} + (M_{G_{ya}})\,\overline{j} + (M_{G_{za}})\,\overline{k}$$

$$M_{G_{xa}} = \left[-Y_{aM}\,(\tau_{Lza} - \tau_{Rza}) - h_L\,\tau_{Lya} - h_R\,\tau_{Rya} - h_N\,\tau_{nya}\right]$$

$$M_{G_{ya}} = \left[X_{aM}\,(\tau_{Lza} + \tau_{Rza}) + h_L\,\tau_{Lxa} + h_R\,\tau_{rxa} - X_{aN}\,\tau_{Nza} + h_N\,\tau_{Nxa}\right]$$

$$M_{G_{za}} = \left[-X_{aM}\,(\tau_{Lya} + \tau_{Rya}) + Y_{aM}\,(\tau_{Lxa} - \tau_{Rxa}) + X_{aN}\,\tau_{Nya}\right]$$

EVALUATION OF LANDING GEAR GROUND REACTION COMPONENTS

In general the three landing gear components are:

$\mathcal{T}_{1i} \Rightarrow$ along wheel plane trace in X_e, Y_e plane.

$\mathcal{T}_{2i} \Rightarrow$ in X_e, Y_e plane and perpendicular to wheel plane trace.

$\mathcal{T}_{3i} \Rightarrow$ in Z_e direction.

where i = L.R.N.

See Figure 4 for sketch of related axes systems.

It is assumed $\mathcal{T}_{1i}$ are due wheel braking, wheel "spin-up", friction about wheel axle and friction between tire and runway.

$\mathcal{T}_{2i}$ are due to tire sideslip.

$\mathcal{T}_{3i}$ are due to landing gear strut axial loads.

THIRD COMPONENTS $\mathcal{T}_{3i}$:

By assumption 6, the landing gear strut line of action is perpendicular to the X_a, Y_a plane. Consequently, the landing gear thrust load applied by the gear to the ground can be represented in the body axes system by

$$\bar{\mathcal{T}}_i = \bar{\mathcal{T}}_i \bar{k}, \; i = L, R, N$$

It is assumed that the ground reacts to only that component of $\mathcal{T}_i$ normal to the ground plane. Components of $\mathcal{T}_i$ in the ground plane are assumed to contribute only to bending of the landing gear.

The vertical ground reaction is $\mathcal{T}_{3i}$ which as a vector is:

$$(\mathcal{T}_{3i})\,\bar{n}$$

$$\therefore (\bar{\tau}_i) \cdot \hat{n} + \tau_{3i} = 0$$

$$\tau_{3i} = -\tau_i \, (\bar{k} \cdot \hat{n})$$

From the transfer equations body to $\hat{X}_e$, $\hat{Y}_e$, $\hat{Z}_e$ system on page A8-17:

$$\bar{k} = \left[\cos\phi \sin\theta \, \hat{S} - \sin\phi \, \hat{t} + \cos\phi \cos\theta \, \hat{n}\right]$$

$\hat{n}$ as components in $\hat{X}_e$, $\hat{Y}_e$ $\hat{Z}_e$ system is:

$$\hat{n} = \left[(0)\,\hat{S} + (0)\,\hat{t} + (1)\,\hat{n}\right]$$

$$\therefore (\bar{k} \cdot \hat{n}) = \cos\phi \cos\theta$$

and

$$\tau_{3i} = -\tau_i \, (\cos\phi \cos\theta)$$

or
$$\tau_{3N} = -\tau_N \, (\cos\phi \cos\theta)$$
$$\tau_{3L} = -\tau_L \, (\cos\phi \cos\theta)$$
$$\tau_{3R} = -\tau_P \, (\cos\phi \cos\theta)$$

The determination of τ_N, τ_R, τ_L follows.

A possible analogue of an aircraft landing gear is as follows:

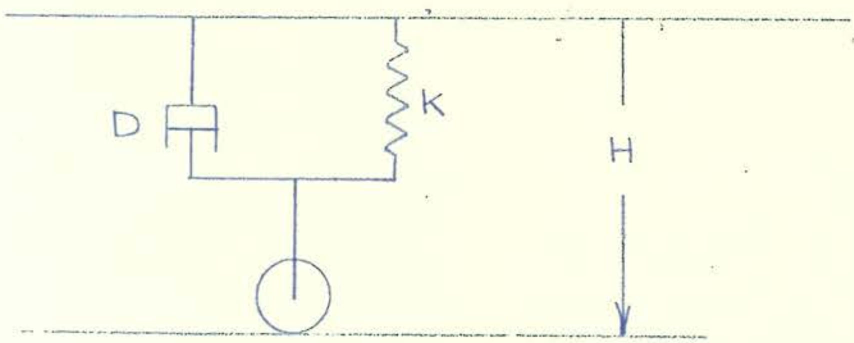


Figure 5

Where "D" is shock strut dashpot constant, "K" is shock strut spring constant. "H" is the maximum length of the unloaded gear. Let h_g represent actual extension of gear under load and let τ represent the ground reaction.

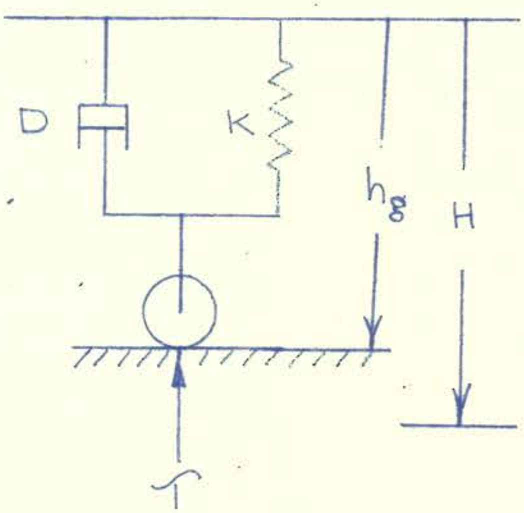


Figure 6

Figure 6 can be replaced by the following force diagram.

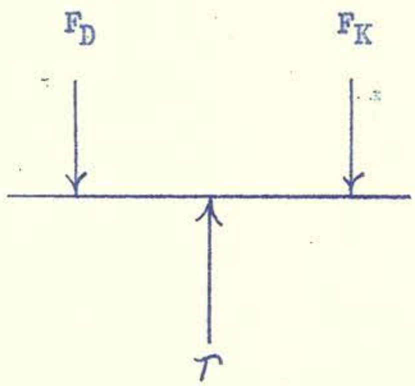


Figure 7

Now:

$$F_D = -\,Dh_g$$

$$F_K = Kh_g$$

The <u>magnitude</u> of the ground reaction force is given by:

$$\tau = -Dh_g + Kh_g$$

By consequence of assumption 6, in vector form the ground reaction is:

$$\tau = -\tau \bar{k}$$

But since the ground is a "one way" restraint:

$$\tau = -D\dot{h}_g + Kh_g \geq 0$$

GROUND REACTION FORCES DUE TO GEAR ARE:

$\tau_L = -D_L\dot{h}_L + K_M h_L$ left main gear

$\tau_R = -D_R\dot{h}_R + K_M h_R$ right main gear

$\tau_N = -D_N\dot{h}_N + K_N h_n$ nose gear

where subscript

N $\Longrightarrow$ nose gear

R $\Longrightarrow$ right gear

L $\Longrightarrow$ left gear

Evaluation of h_R, h_L, h_n:

Consider case: $\phi = 0$, ϕ-body axes Euler roll angle.

$\theta \neq 0$, θ - body axes Euler pitch angle.

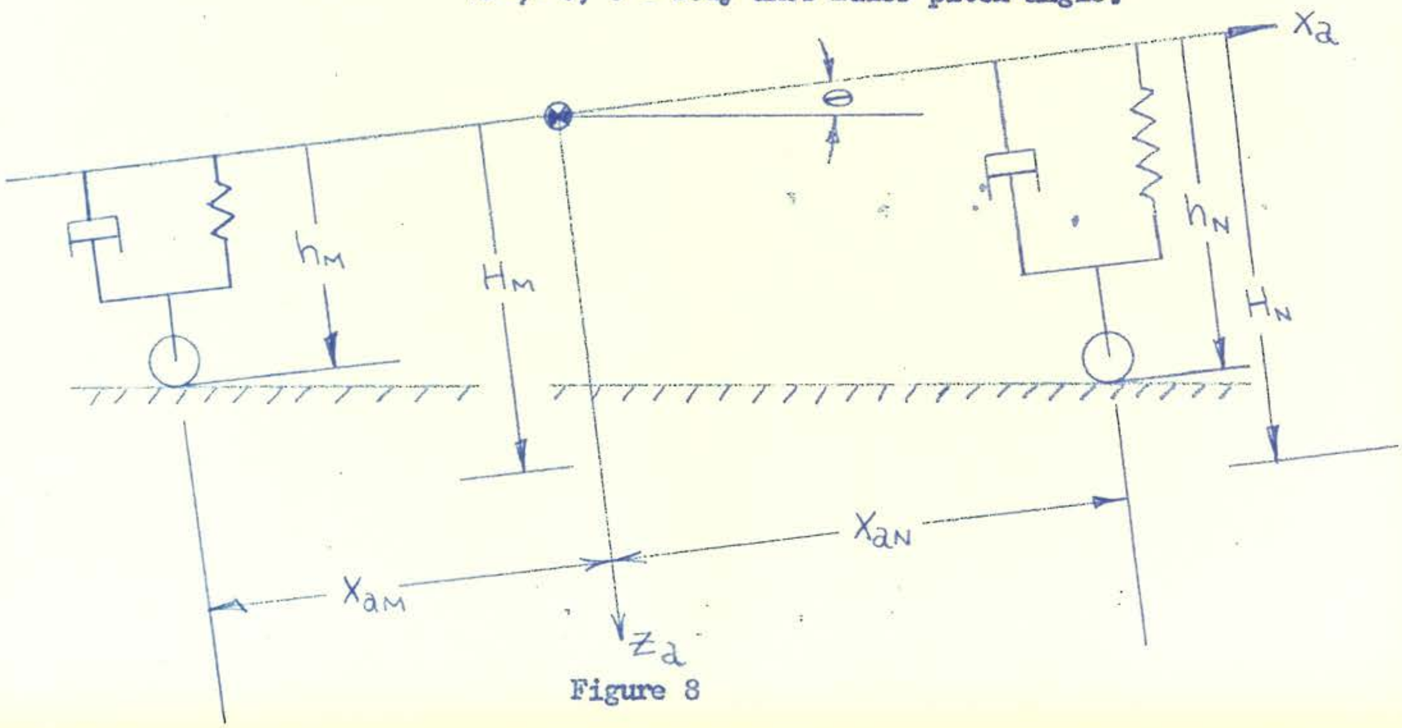
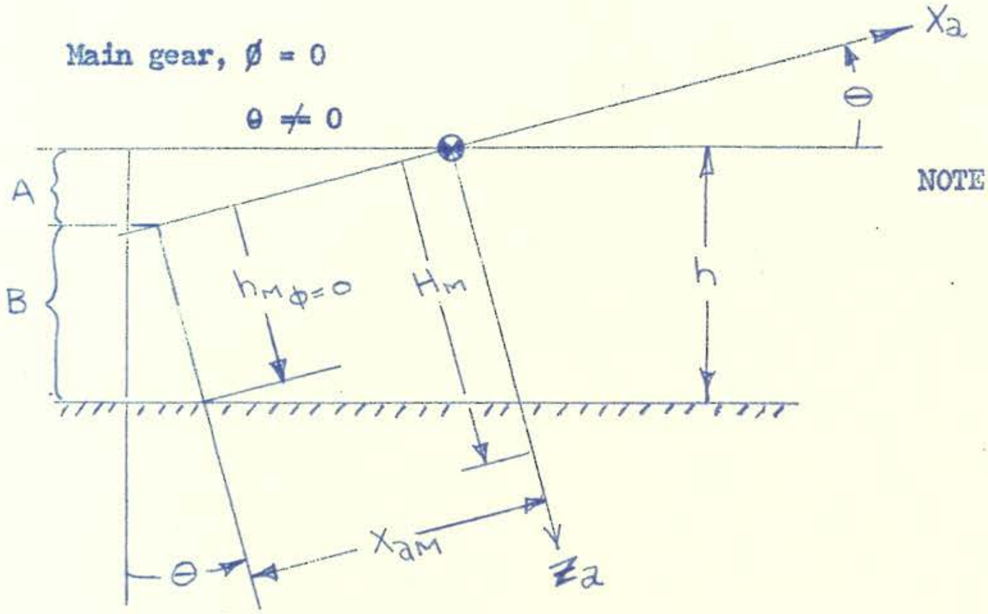


Figure 8



NOTE: h ⟹ altitude CG above ground

X_{aM} ⟹ absolute value X_a coordinate of main gear line of action intersection with X_a, Y_a plane.

$A = X_{aM} \sin\theta$
$B = h - A = h - X_{aM} \sin\theta$

Figure 9

$$\therefore h_{M\phi=0} = \frac{B}{\cos\theta} = \left[\frac{h}{\cos\theta} - X_{am} \tan\theta\right], \quad h < H_m \cos\theta + X_{am} \sin\theta$$

$$h_{M\phi=0} = H_M, \quad h \geq H_M \cos\theta + X_{am} \sin\theta$$

$$h_{M\phi=0} = h_{R\phi=0} = h_{L\phi=0}$$

$$H_{m_o} \leq h_R,\ h_L \leq H_m$$

H_{m_o} ⟹ Bottomed landing gear.

Nose gear, $\phi = 0$

$\theta \neq 0$

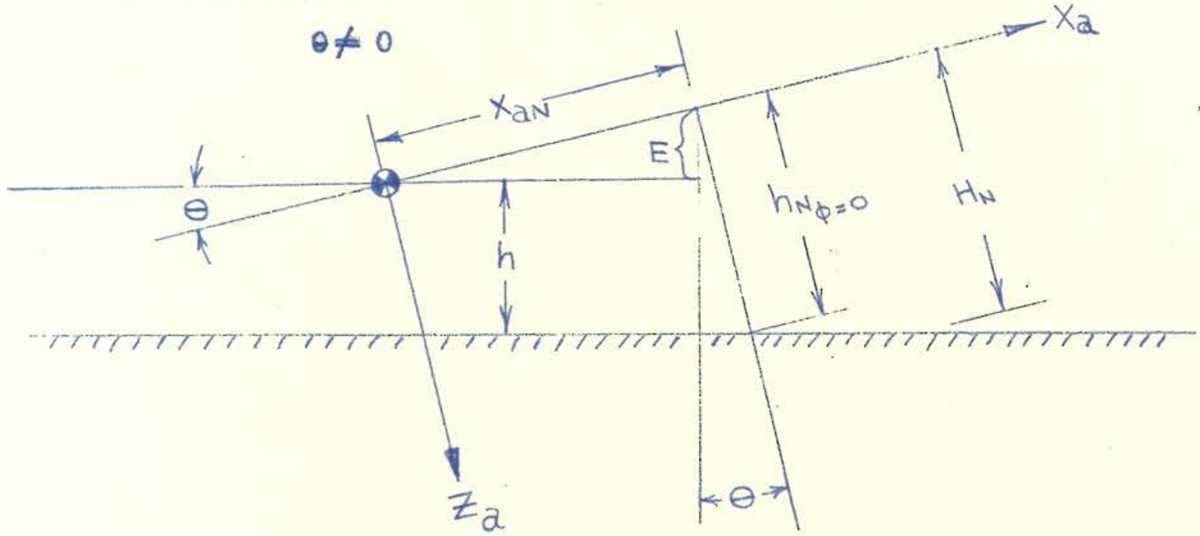


Figure 10

NOTE: h ⇒ altitude CG above ground

X_{aN} ⇒ Absolute value X_a coordinate of nose gear line of action intersection with X_a axis.

$$E = X_{aN} \sin \theta$$

$$\therefore h_{N\phi=0} = \left[\frac{h}{\cos \theta} + X_{aN} \tan \theta\right], \quad h < \left[H_N \cos \theta - X_{aN} \sin \theta\right]$$

$$h_{N\phi=0} = H_N \qquad h \geq \left[H_N \cos \theta - X_{aN} \sin \theta\right]$$

$$H_{N_o} \geq h_N, \; h_{N\phi=0} \geq H_N$$

H_{N_o} ⇒ Bottomed nose gear

Main Gear $\phi \neq 0$

$\theta \neq 0$

Consider aircraft with pitch angle θ and roll angle ϕ. Then looking at projection on plane that both contains line of action of main gear and is perpendicular to X_a axis.

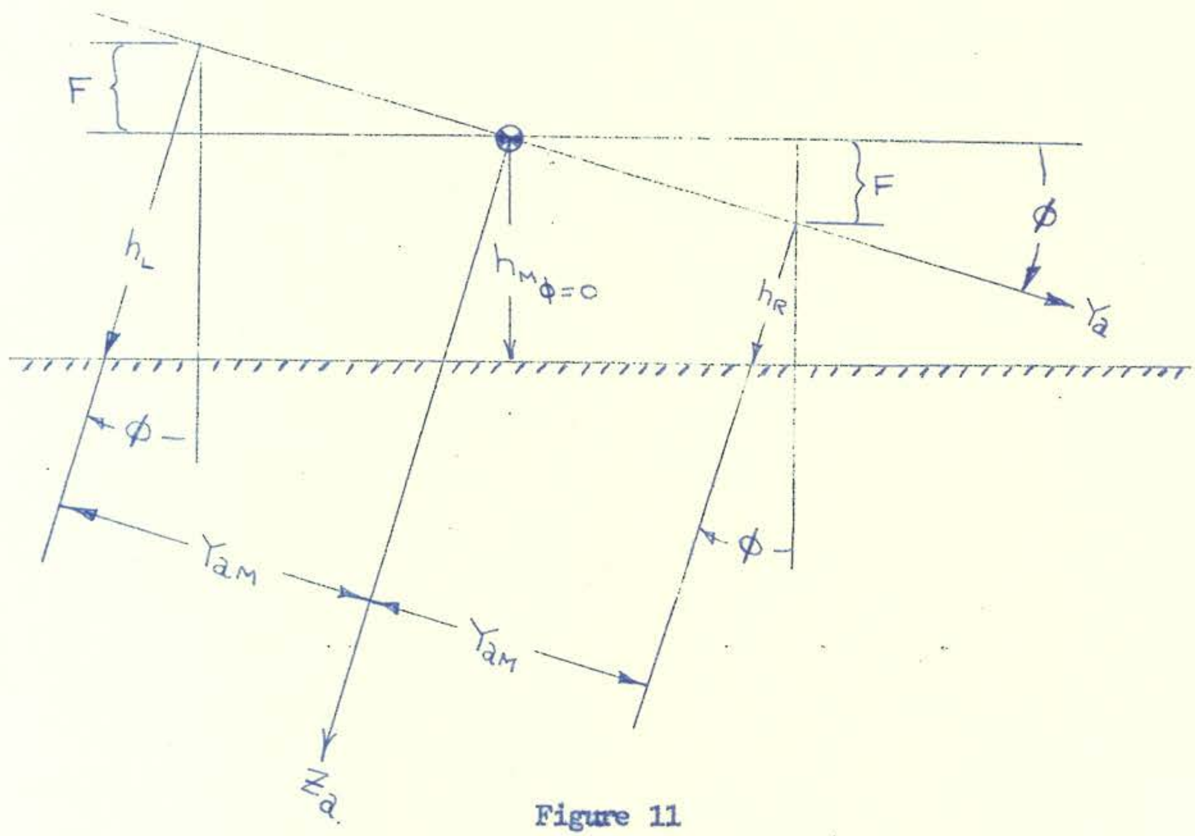


Figure 11

NOTE: $Y_{aM} \Rightarrow$ absolute value

Y_a coordinate of main gear line of action intersection with X_a, Y_a plane.

$$F = Y_{aM} \sin \phi$$

$$h_R \cos \phi = h_{m\phi=0} - F$$

$$h_L \cos \phi = h_{m\phi=0} + F$$

$$h_R = \left[\frac{h}{\cos\theta\cos\phi} - X_{aM}\frac{\tan\theta}{\cos\phi} - Y_{aM}\tan\phi\right] \quad h < \left[H\cos\theta\cos\phi + X_{aM}\sin\theta + Y_{aM}\cos\theta\sin\phi\right],$$

$$h_R = H_M, \quad h \geq \left[H_M\cos\theta\cos\phi + X_{aM}\sin\theta + Y_{aM}\cos\theta\sin\phi\right]$$

$$h_L = \left[\frac{h}{\cos\theta\cos\phi} - \frac{X_{aM}\tan\theta}{\cos\phi} + Y_{aM}\tan\phi\right], \quad h < \left[H_M\cos\theta\cos\phi + X_{aM}\sin\theta - Y_{aM}\cos\theta\sin\phi\right]$$

$$h_L = H_M, \quad h \geq \left[H_M\cos\theta\cos\phi + X_{aM}\sin\theta - Y_{aM}\cos\theta\sin\phi\right]$$

Nose gear $\phi \neq 0$

$\theta \neq 0$

Consider aircraft with pitch angle θ and roll angle ϕ. Then looking at projection on plane that both contains line of action of nose gear and is perpendicular to X_a axis.

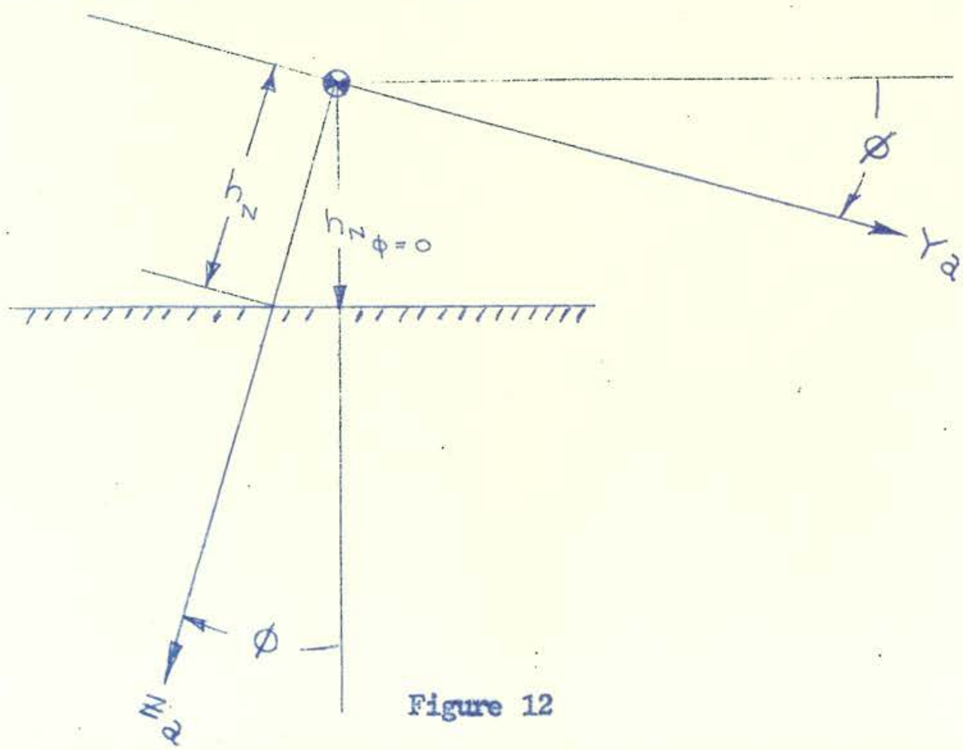


Figure 12

$$h_N = \left[\frac{h}{\cos\theta\cos\phi} + X_{aN}\frac{\tan\theta}{\cos\phi}\right],\ h < \left[H_N\cos\theta\cos\phi - X_{aN}\sin\theta\right]$$

$$h_N = H_N,\ h \geq \left[H_N\cos\theta\cos\phi - X_{aN}\sin\theta\right]$$

SUMMARY OF LANDING GEAR EXTENSIONS EQUATIONS FOR $\phi \neq 0,\ \theta \neq 0$:

NOSE GEAR:

$$h_N = \left[\frac{h}{\cos\theta\cos\phi} + X_{aN}\frac{\tan\theta}{\cos\phi}\right],\ h < \left[H_N\cos\theta\cos\phi - X_{aN}\sin\theta\right]$$

$$h_N = H_N \qquad ,\ h \geq \left[H_N\cos\theta\cos\phi - X_{aN}\sin\theta\right]$$

$$H_{N_o} \leq h_N \leq H_N$$

RIGHT MAIN GEAR:

$$h_R = \left[\frac{h}{\cos\theta\cos\phi} - X_{aM}\frac{\tan\theta}{\cos\phi} - Y_{aM}\tan\phi\right],\ h < \left[H_M\cos\theta\cos\phi + X_{aM}\sin\theta + Y_{aM}\cos\theta\sin\phi\right]$$

$$h_R = H_M \qquad ,\ h \geq \left[H_M\cos\theta\cos\phi + X_{aM}\sin\theta + Y_{aM}\cos\theta\sin\phi\right]$$

$$H_{M_o} \leq h_R \leq H_M$$

LEFT MAIN GEAR:

$$h_L = \left[\frac{h}{\cos\theta\cos\phi} - X_{aM}\frac{\tan\theta}{\cos\phi} + Y_{aM}\tan\phi\right],\ h < \left[H_M\cos\theta\cos\phi + X_{aM}\sin\theta - Y_{aM}\cos\theta\sin\phi\right]$$

$$h_L = H_M \qquad ,\ h \geq \left[H_M\cos\theta\cos\phi + X_{aM}\sin\theta - Y_{aM}\cos\theta\sin\phi\right]$$

$$H_{M_o} \leq h_L \leq H_M$$

$$\dot{h}_N = \frac{d}{dt}\left[\frac{h}{\cos\theta\cos\phi}\right] + \frac{d}{dt}\left[X_{aN}\frac{\tan\theta}{\cos\phi}\right]$$

$$\dot{h}_R = \frac{d}{dt}\left[\frac{h}{\cos\theta\cos\phi}\right] - \frac{d}{dt}\left[X_{aM}\frac{\tan\theta}{\cos\phi}\right] - \frac{d}{dt}\left[Y_{aM}\tan\phi\right]$$

$$\dot{h}_L = \frac{d}{dt}\left[\frac{h}{\cos\theta\cos\phi}\right] - \frac{d}{dt}\left[X_{aM}\frac{\tan\theta}{\cos\phi}\right] + \frac{d}{dt}\left[Y_{aM}\tan\phi\right]$$

Assuming time rate of change of aircraft C.G. position is insignificantly small, then X_{aN}, X_{aM}, Y_{aM} will act as constants during differentiation with respect to time.

$$\frac{d}{dt}\left[\frac{h}{\cos\theta\cos\phi}\right] = \frac{\cos\theta\cos\phi\frac{dh}{dt} + h\left[\cos\theta\sin\phi\frac{d\phi}{dt} + \sin\theta\cos\phi\frac{d\theta}{dt}\right]}{\cos^2\theta\cos^2\phi}$$

$$\frac{d}{dt}\left[\frac{h}{\cos\theta\cos\phi}\right] = \frac{\dot{h}}{\cos\theta\cos\phi} + \frac{h\dot{\phi}\tan\phi}{\cos\theta\cos\phi} + \frac{h\dot{\theta}\tan\theta}{\cos\theta\cos\phi}$$

$$\frac{d}{dt}\left[\frac{\tan\theta}{\cos\phi}\right] = \frac{d}{dt}\left[\frac{\sin\theta}{\cos\theta\cos\phi}\right] = \frac{\cos^2\theta\cos\phi\frac{d\theta}{dt} + \sin\theta\left[\cos\theta\sin\phi\frac{d\phi}{dt} + \sin\theta\cos\phi\frac{d\theta}{dt}\right]}{\cos^2\theta\cos^2\phi}$$

$$\frac{d}{dt}\left[\frac{\tan\theta}{\cos\phi}\right] = \frac{\dot{\theta}\cos\theta}{\cos\theta\cos\phi} + \frac{\dot{\theta}\sin\theta\tan\theta}{\cos\theta\cos\phi} + \frac{\dot{\phi}\sin\theta\tan\phi}{\cos\theta\cos\phi}$$

$$\frac{d}{dt}\left[\frac{\tan\theta}{\cos\phi}\right] = \frac{\dot{\phi}\tan\theta\tan\phi}{\cos\phi} + \frac{\dot{\theta}\sec^2\theta}{\cos\phi}$$

$$\frac{d}{dt}\left[\tan\phi\right] = \frac{d}{dt}\left[\frac{\sin\phi}{\cos\phi}\right] = \frac{\cos\phi\cos\phi\frac{d\phi}{dt} + \sin\phi\sin\phi\frac{d\phi}{dt}}{\cos^2\phi}$$

$$\frac{d}{dt}\left[\tan\phi\right] = \dot{\phi}\sec^2\phi$$

SUMMARY LANDING GEAR STRUT LOADS

$$h_N = \frac{h}{\cos\theta\cos\phi} + \frac{h\dot{\phi}\tan\phi}{\cos\theta\cos\phi} + \frac{h\dot{\theta}\tan\theta}{\cos\theta\cos\phi} + X_{aN}\dot{\phi}\frac{\tan\theta\tan\phi}{\cos\phi} + X_{aN}\frac{\dot{\theta}\sec^2\theta}{\cos\phi}$$

$$\dot{h}_R = \frac{\dot{h}}{\cos\theta\cos\phi} + \frac{h\dot{\phi}\tan\phi}{\cos\theta\cos\phi} + \frac{h\dot{\theta}\tan\theta}{\cos\theta\cos\phi} - X_{aM}\dot{\phi}\frac{\tan\theta\tan\phi}{\cos\phi} - X_{aM}\dot{\theta}\frac{\sec^2\theta}{\cos\phi} - Y_{aM}\dot{\phi}\sec^2\phi$$

$$\dot{h}_L = \frac{\dot{h}}{\cos\theta\cos\phi} + \frac{h\,\dot{\phi}\tan\phi}{\cos\theta\cos\phi} + \frac{h\dot{\theta}\tan\theta}{\cos\theta\cos\phi} - X_{a_M}\,\dot{\phi}\,\frac{\tan\theta\tan\phi}{\cos\phi}$$
$$- X_{a_M}\,\dot{\theta}\,\frac{\sec^2\theta}{\cos\phi} + Y_{a_M}\,\dot{\phi}\sec^2\phi$$

$$h_N = \left[\frac{h}{\cos\theta\cos\phi} + X_{a_N}\frac{\tan\theta}{\cos\phi}\right], \quad H_{N_o} \le h_N \le H_N$$

$$h_R = \left[\frac{h}{\cos\theta\cos\phi} - X_{a_M}\frac{\tan\theta}{\cos\phi} - Y_{a_M}\tan\phi\right], \quad H_{M_o} \le h_R \le H_M$$

$$h_L = \left[\frac{h}{\cos\theta\cos\phi} - X_{a_M}\frac{\tan\theta}{\cos\phi} + Y_{a_M}\tan\phi\right], \quad H_{M_o} \le h_L \le H_M$$

$$\mathcal{T}_N = -D_N\,\dot{h}_N + K_N\,h_N, \quad \mathcal{T}_N \ge 0$$

$$\mathcal{T}_R = -D_R\,\dot{h}_R + K_R\,h_R, \quad \mathcal{T}_R \ge 0$$

$$\mathcal{T}_L = -D_L\,\dot{h}_L + K_L\,h_L, \quad \mathcal{T}_L \ge 0$$

Second Components - $\mathcal{T}_{2i}$

According to Douglas Report DS-1913-LA, pages 3.2.1.2 and 3.2.4.1.10, tire side forces are given by:

$$\mathcal{T}_{2i} = -R_i\,f(B_i), \quad i = L,R,N$$

B_i = tire slip angle

In our notation:

$$R_i = -\mathcal{T}_{3i}$$

$f(B_i) = C_{SF}$, tire side force coefficient

$\therefore \mathcal{T}_{2i} = \mathcal{T}_{3i}\,(C_{SF})_i$, where: $\mathcal{T}_{3i} \le 0$

$C_{SF_i} > 0,\ B_i > 0$

$\therefore B_i > 0,\ \mathcal{T}_{2i} < 0$

where $(C_{SF})_i = f(B_i$, runway conditions)

$$B_i = \tan^{-1}\left[\frac{V_{Y_{ei}}}{V_{X_{ei}}}\right]$$

$V_{X_{ei}}, V_{Y_{ei}} \Longrightarrow$ Respective projections on appropriate "trace axes system" of absolute velocity vector of wheel center of rotation.

VELOCITY WITH RESPECT TO GROUND PLANE OF POINT OF INTERSECTION WHEEL AXLE AND WHEEL PLANE:

In general, for a wheel on aircraft:

$$\bar{V}_W/GP = V_{xe}\,\hat{s} + V_{ye}\,\hat{t} + (\bar{\omega} \times \bar{R}_W)$$

$V_{xe}, V_{ye} \Longrightarrow$ Respective projections on $\hat{X}_e, \hat{Y}_e$ axes of absolute velocity vector of CG.

$\bar{\omega} \Longrightarrow$ Rotational velocity vector of aircraft.

$\bar{R}_W \Longrightarrow$ Position vector in body axes system of point of intersection of wheel axle with wheel plane.

Left main gear;

$$\bar{R}_L = (-X_{aM})\bar{i} + (-Y_{aM})\bar{j} + (h_L)\bar{k}$$

$$\bar{\omega}\times\bar{R}_L = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ p_a & q_a & r_a \\ (-X_{aM}) & (-Y_{aM}) & (h_L) \end{vmatrix} = \begin{matrix} \left[h_L\, q_a + Y_{aM}\, r_a\right]\bar{i} \\ -\left[h_L\, p_a + X_{aM}\, r_a\right]\bar{j} \\ \left[X_{aM}\, q_a - Y_{aM}\, p_a\right]\bar{k} \end{matrix}$$

From page A8-13 of AIRCRAFT GROUND REACTIONS

$$\bar{i} = \cos\theta\,\hat{s} - \sin\theta\,\hat{n}$$

$$\bar{j} = \sin\phi\sin\theta\,\hat{s} + \cos\phi\,\hat{t} + \sin\phi\cos\theta\,\hat{n}$$

$$\bar{k} = \cos\phi\sin\theta\,\hat{s} - \sin\phi\,\hat{t} + \cos\phi\cos\theta\,\hat{n}$$

$$\bar{\omega} \times \bar{R}_L = \Big[(h_L\, q_a + Y_{aM}\, r_a) \cos \theta - (h_L\, p_a + X_{aM}\, r_a) \sin \phi \sin \theta + (X_{aM}\, q_a - Y_{aM}\, p_a) \cos \phi \sin \theta\Big] \hat{s}$$

$$- \Big[(h_L\, p_a + X_{aM}\, r_a) \cos \phi + (X_{aM}\, q_a - Y_{aM}\, p_a) \sin \phi\Big] \hat{t}$$

$$- \Big[(h_L\, q_a + Y_{aM}\, r_a) \sin \theta + (h_L\, p_a + X_{aM}\, r_a) \sin \phi \cos \theta - (X_{aM}\, q_a - Y_{aM}\, p_a) \cos \phi \cos \theta\Big] \hat{n}$$

And of which we're interested only in $\hat{s}$, $\hat{t}$ components.

Left Main Gear:

$$\therefore \bar{V}_L/GP = \Big[V_{xe} + (h_L\, q_a + Y_{aM}\, r_a) \cos \theta - (h_L\, p_a + X_{aM}\, r_a) \sin \phi \sin \theta + (X_{aM}\, q_a - Y_{aM}\, p_a) \cos \phi \sin \theta\Big] \hat{s}$$

$$+ \Big[V_{Y_e} - (h_L\, p_a + X_{aM}\, r_a) \cos \phi - (X_{aM}\, q_a) - Y_{aM}\, p_a) \sin \phi\Big] \hat{t}$$

From page A8-16:

$$\bar{T}_1 = \cos \psi_{PS}\, \hat{s} + \sin \psi_{PS}\, \hat{t}$$

$$\bar{T}_2 = - \sin \psi_{PS}\, \hat{s} + \cos \psi_{PS}\, \hat{t}$$

$$\bar{T}_3 = \hat{n}$$

$$\therefore \bar{V}_L/GP = \Big[V_{X_e} \cos \psi_{PS} + (h_L\, q_a + Y_{aM}\, r_a) \cos \theta \cos \psi_{PS} - (h_L\, p_a + X_{aM}\, r_a) \sin \phi \sin \theta \cos \psi_{PS} + (X_{aM}\, q_a - Y_{aM}\, p_a) \cos \phi \sin \theta \cos \psi_{PS} + V_{Y_e} \sin \psi_{PS} - (h_L\, p_a + X_{aM}\, r_a) \cos \phi \sin \psi_{PS} - (X_{aM}\, q_a - Y_{aM}\, p_a) \sin \phi \sin \psi_{PS}\Big] \bar{T}_1$$

$$\Big[- V_{X_e} \sin \psi_{PS} - (h_L\, q_a + Y_{aM}\, r_a) \cos \theta \sin \psi_{PS} + (h_L\, p_a + X_{aM}\, r_a) \sin \phi \sin \theta \sin \psi_{PS} - (X_{aM}\, q_a - Y_{aM}\, p_a) \cos \phi \sin \theta \sin \psi_{PS} + V_{Y_e} \cos \psi_{PS} - (h_L\, p_a + X_{aM}\, r_a) \cos \phi \cos \psi_{PS} - (X_{aM}\, q_a - Y_{aM}\, p_a) \sin \phi \cos \psi_{PS}\Big] \bar{T}_2$$

For right main gear:

$$\bar{R}_R = (-X_{aM})\,\bar{i} + (Y_{aM})\,\bar{j} + (h_R)\,\bar{k}$$

For nose gear:

$$\bar{R}_N = (X_{aN})\,\bar{i} + (h_N)\,\bar{k}$$

and the trace angle is ψ_{NP}.

$$\therefore \bar{V}_{R/GP} = \Big[V_{X_e} \cos\psi_{PS} + (h_R\, q_a - Y_{aM}\, r_a) \cos\theta \cos\psi_{PS} - (h_R\, p_a + X_{aM}\, r_a) \sin\phi \sin\theta \cos\psi_{PS} + (X_{aM}\, q_a + Y_{aM}\, p_a) \cos\phi \sin\theta \cos\psi_{PS} + V_{Y_e} \sin\psi_{PS} - (h_R\, p_a + X_{aM}\, r_a) \cos\phi \sin\psi_{PS} - (X_{aM}\, q_a + Y_{aM}\, p_a) \sin\phi \sin\psi_{PS}\Big]\,\bar{T}_1$$

$$\Big[- V_{X_e} \sin\psi_{PS} - (h_R\, q_a - Y_{aM}\, r_a) \cos\theta \sin\psi_{PS} + (h_R\, p_a + X_{aM}\, r_a) \sin\phi \sin\theta \sin\psi_{PS} - (X_{aM}\, q_a + Y_{aM}\, p_a) \cos\phi \sin\theta \sin\psi_{PS} + V_{Y_e} \cos\psi_{PS} - (h_R\, p_a + X_{aM}\, r_a) \cos\phi \cos\psi_{PS} - (X_{aM}\, q_a + Y_{aM}\, p_a) \sin\phi \cos\psi_{PS}\Big]\,\bar{T}_2$$

$$\bar{V}_{N/GP} = \Big[V_{X_e} \cos\psi_{NP} + (h_N\, q_a) \cos\theta \cos\psi_{NP} - (h_N\, p_a - X_{aN}\, r_a) \sin\phi \sin\theta \cos\psi_{NP} - (X_{aN}\, q_a) \cos\phi \sin\theta \cos\psi_{NP} + V_{Y_e} \sin\psi_{NP} - (h_N\, p_a - X_{aN}\, r_a) \cos\phi \sin\psi_{NP} + (X_{aN}\, q_a) \sin\phi \sin\psi_{NP}\Big]\,\bar{N}_1$$

$$\Big[- V_{X_e} \sin\psi_{NP} - (h_N\, q_a) \cos\theta \sin\psi_{NP} + (h_N\, p_a - X_{aN}\, r_a) \sin\phi \sin\theta \sin\psi_{NP} + (X_{aN}\, q_a) \cos\phi \sin\theta \sin\psi_{NP} + V_{Y_e} \cos\psi_{NP} - (h_N\, p_a - X_{aN}\, r_a) \cos\phi \cos\psi_{NP} + (X_{aN}\, q_a) \sin\phi \cos\psi_{NP}\Big]\,\bar{N}_2$$

First Simplifications:

Let θ, ϕ, ψ_{PS} be small angles; neglect second order effect, i.e., $(\theta\phi)$

$$\overline{V}_L/GP = \Big[V_{X_e} + (h_L\, q_a + Y_{aM}\, r_a) + (X_{aM}\, q_a - Y_{aM}\, p_a)\, \theta + V_{Y_e}\, \psi_{PS} + (h_L\, p_a + X_{aM}\, r_a)\, \psi_{PS}\Big]\, \overline{T}_1$$

$$+ \Big[- V_{X_e}\, \psi_{PS} - (h_L\, q_a + Y_{aM}\, r_a)\, \psi_{PS} + V_{Y_e} - (h_L\, p_a + X_{aM}\, r_a) - (X_{aM} q_a - Y_{aM}\, p_a)\, \phi\Big]\, \overline{T}_2$$

$$\overline{V}_R/GP = \Big[V_{X_e} + (h_R\, q_a - Y_{aM}\, r_a) + (X_{aM}\, q_a + Y_{aM}\, p_a)\, \theta + V_{Y_e}\, \psi_{PS} - (h_R\, p_a + X_{aM}\, r_a)\, \psi_{PS}\Big]\, T_1$$

$$+ \Big[- V_{X_e}\, \psi_{PS} - (h_R\, q_a - Y_{aM}\, r_a)\, \psi_{PS} + V_{Y_e} - (h_R\, p_a + X_{aM}\, r_a) - (X_{aM}\, q_a + Y_{aM}\, p_a)\, \phi\Big]\, \overline{T}_2$$

$$\overline{V}_N/GP = \Big[V_{X_e} \cos\psi_{NP} + (h_N\, q_a) \cos\psi_{NP} + (X_{aN}\, q_a)\, (\phi \sin\psi_{NP} - \theta \cos\psi_{NP}) + V_{Y_e} \sin\psi_{NP} - (h_N\, p_a - X_{aN}\, r_a) \sin\psi_{NP}\Big]\, \overline{N}_1$$

$$\Big[- V_{X_e} \sin\psi_{NP} - (h_N\, q_a) \sin\psi_{NP} + (X_{aN}\, q_a)\, (\phi \cos\psi_{NP} + \theta \sin\psi_{NP}) + V_{Y_e} \cos\psi_{NP} - (h_N\, p_a - X_{aN}\, r_a) \cos\psi_{NP}\Big]\, \overline{N}_2$$

Second Simplification:

Let $\psi_{PS} = 0$

$$\overline{V}_L/GP = \Big[V_{X_e} + (h_L\, q_a + Y_{aM}\, r_a) + (X_{aM}\, q_a - Y_{aM}\, q_a)\, \theta\Big]\, \overline{T}_1$$

$$\Big[V_{Y_e} - (h_L\, p_a + X_{aM}\, r_a) - (X_{aM}\, q_a - Y_{aM}\, p_a)\, \phi\Big]\, \overline{T}_2$$

$$\overline{V}_R/GP = \Big[V_{X_e} + (h_R\, q_a - Y_{aM}\, r_a) + (X_{aM}\, q_a + Y_{aM}\, p_a)\, \theta\Big]\, \overline{T}_1$$

$$\Big[V_{Y_e} - (h_R\, p_a + X_{aM}\, r_a) - (X_{aM}\, q_a + Y_{aM}\, p_a)\, \phi\Big]\, \overline{T}_2$$

$$\tan\psi_{NP} = \Big[- \tan\phi \sin\theta + \tan\lambda_N \frac{\cos\theta}{\cos\phi}\Big] \approx \tan\lambda_N$$

$$\overline{V}_N/GP = \Big[V_{X_e} \cos\lambda_N + (h_N\, q_a) \cos\lambda_N + (X_{aN}\, q_a)\, (\phi \sin\lambda_N - \theta \cos\lambda_N) + V_{Y_e} \sin\lambda_N - (h_N\, p_a - X_{aN}\, r_a) \sin\lambda_N\Big]\, \overline{N}_1$$

$$\Big[- V_{X_e} \sin\lambda_N - (h_N\, q_a) \sin\lambda_N + (X_{aN}\, q_a)\, (\phi \cos\lambda_N + \theta \sin\lambda_N)$$
$$+ V_{Y_e} \cos\lambda_N - (h_N\, p_a - X_{aN}\, r_a) \cos\lambda_N\Big]\, \overline{N}_2$$

Third Simplification:

At touchdown when measured in radians $\theta \approx 0$, $\phi \approx 0$:

$$\overline{V}_L/GP = \left[V_{X_e} + (h_L\, q_a + Y_{aM}\, r_a)\right] \overline{T}_1 + \left[V_{Y_e} - (h_L\, p_a + X_{aM}\, r_a)\right] \overline{T}_2$$

$$\overline{V}_R/GP = \left[V_{X_e} + (h_R\, q_a - Y_{aM}\, r_a)\right] \overline{T}_1 + \left[V_{Y_e} - (h_R\, p_a + X_{aM}\, r_a)\right], \overline{T}_2$$

$$\overline{V}_N/GP = \left\{(V_{X_e} + h_N\, q_a) \cos\lambda_N + \left[V_{Y_e} - (h_N\, p_a - X_{aN}\, r_a)\right] \sin\lambda_N\right\} \overline{N}_1$$
$$+ \left\{\left[V_{Y_e} - (h_N\, p_a - X_{aN}\, r_a)\right] \cos\lambda_N - (V_{X_e} + h_N\, q_a) \sin\lambda_N\right] \overline{N}_2$$

Fourth Simplification:

Let $h_R = h_L = h_N = h_M$ = an average value of landing gear extension:

$$\overline{V}_L/GP = \left[V_{X_e} + (h_M\, q_a + Y_{aM}\, r_a)\right] \overline{T}_1 + \left[V_{Y_e} - (h_M\, p_a + X_{aM}\, r_a)\right] \overline{T}_2$$

$$\overline{V}_R/GP = \left[V_{X_e} + (h_M\, q_a - Y_{aM}\, r_a)\right] \overline{T}_1 + \left[V_{Y_e} - (h_M\, p_a + X_{aM}\, r_a)\right] \overline{T}_2$$

$$\overline{V}_M/GP = \left\{\left[V_{Y_e} - (h_M\, p_a - X_{aN}\, r_a)\right] \sin\lambda_N - (V_{X_e} + h_M\, q_a) \cos\lambda_N\right\} \overline{N}_1$$
$$+ \left\{\left[V_{Y_e} - (h_M\, p_a - X_{aN}\, r_a)\right] \cos\lambda_N - (V_{X_e} + h_M\, q_a) \sin\lambda_N\right\} N_2$$

Fifth Simplification and New Approach to Nose Wheel Side Slip Angle:

Assuming small pitch and roll angles:

Ground plane components of nose wheel velocity vector tangential and normal to plane of symmetry trace on ground plane.

$$\overline{V}_N/GP = (V_{X_e} + h_M\, q_a)\, \overline{T}_1 + \left[V_{Y_e} - (h_M\, p_a - X_{aN}\, r_a)\right] \overline{T}_2$$

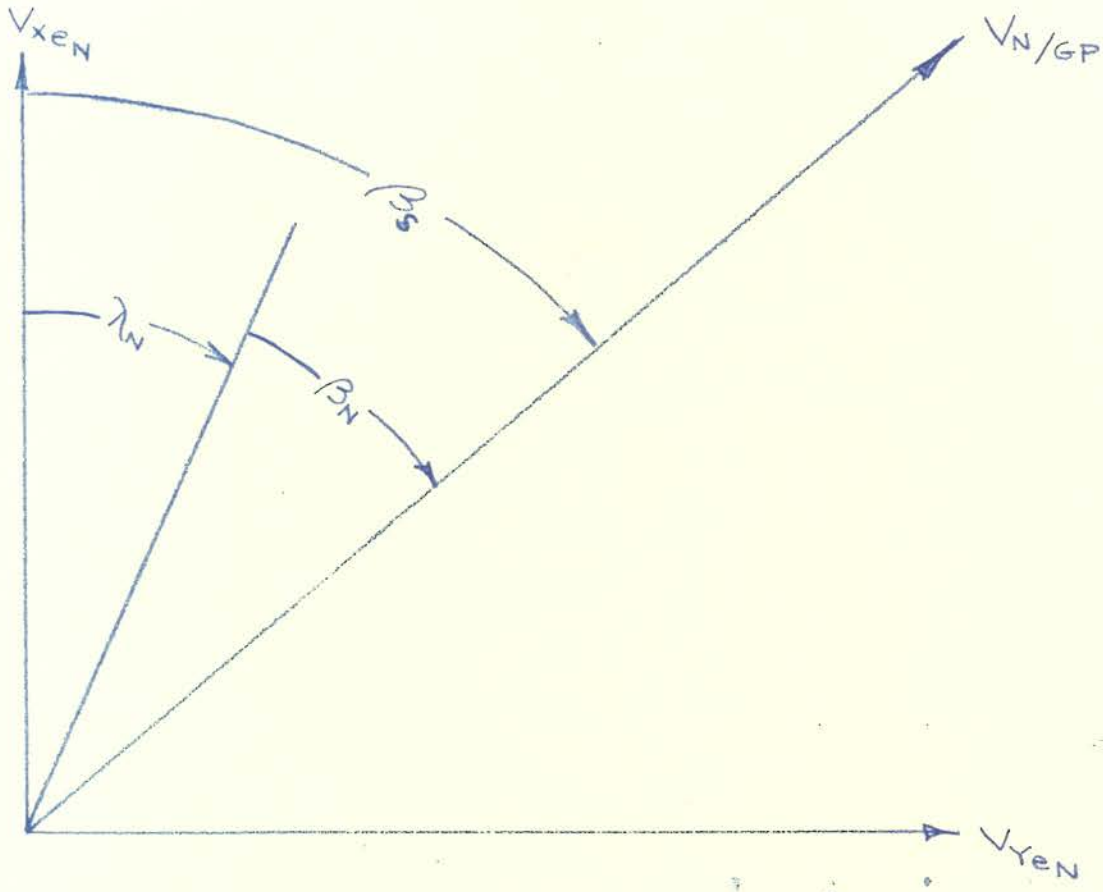


Figure 13

$$\therefore \beta_N = \beta_S - \lambda_N$$

$$\tan \beta_S = \frac{\left[V_{Y_e} - (h_M p_a - X_{aN} r_a)\right]}{(V_{X_e} + h_M q_a)}$$

Fifth Simplification:

$$\overline{V}_L/GP = \left[V_{X_e} + (h_M q_a + Y_{aM} r_a)\right] \overline{T}_1 + \left[V_{Y_e} - (h_M p_a + X_{aM} r_a)\right] \overline{T}_2$$

$$\overline{V}_R/GP = \left[V_{X_e} + (h_M q_a - Y_{aM} r_a)\right] \overline{T}_1 + \left[V_{Y_e} - (h_M p_a + X_{aM} r_a)\right] \overline{T}_2$$

$$\overline{V}_N/GP = \left[V_{X_e} + h_M q_a\right] \overline{T}_1 + \left[V_{Y_e} - (h_M p_a - X_{aN} r_a)\right] \overline{T}_2$$

$$\tan \beta_L = \frac{\left[V_{Y_e} - (h_M p_a + X_{aM} r_a)\right]}{\left[V_{X_e} + (h_M q_a + Y_{aM} r_a)\right]}$$

$$\tan \beta_R = \frac{\left[V_{Y_e} - (h_M p_a + X_{aM} r_a)\right]}{\left[V_{X_e} + (h_M q_a - Y_{aM} r_a)\right]}$$

$$\beta_N = \beta_S - \lambda_N$$

where:

$$\tan \beta_S = \frac{\left[V_{Y_e} - (h_M p_a - X_{aN} r_a)\right]}{\left[V_{X_e} + h_M q_a\right]}$$

First Components $\sim \Upsilon_{1i}$

Υ_{1i} is component contained in the wheel plane and is presumed to arise from wheel braking, rotational friction about wheel axle, wheel rotational accelerations and runway to tire friction characteristics.

According to page 4.3.13 of DACO report DC-1913-1A, the braking contributions to Υ_{1i} are:

$$D_B = K_B \, \delta_{pi} \leq D_{B_{max}} = R_M \mu$$

where:

$K_B \Rightarrow$ shape of curve page 4.3.13 of DS-1913-1A

$\mu \Rightarrow$ curve page A.3i of DS-1913-1A and associated corrections for runway conditions in DACO page PMJ of 10-16-56.

$\delta_{pi} \Rightarrow$ break pedal deflection in degrees.

In our notation:

$$(\Delta \Upsilon_{1i})_B = - K_B \, \delta_{pi}$$

where i = L, R, N

and $\delta_{PN} = 0$.

Rolling friction coefficient is given in addendum of 10-16-56 to page A-3i of DS-1913-1A as:

$\mu_o = .02, \; V_{X_{ei}} \neq 0$

$\mu_o = \text{some value} > .02, \; V_{X_{ei}} = 0$

to simulate breakout friction.

$\therefore$ Rolling friction contributions to τ_{1i} are:

$$(\Delta \tau_{1i})_R = \mu_o \tau_{3i}, \; \tau_{3i} \leq 0, \; \mu_o > 0.$$

It is assumed wheel comes up to speed instantly. Consequently wheel "spin-up" contributions to τ_{1i} are not considered.

$$\tau_{1i} = - K_B \delta_{Pi} + \mu_o \tau_{3i} \geq \tau_{3i} \mu', \; i = L, R, N$$

$$\delta_{PN} = 0$$

DC-8 LANDING GEAR DAMPING COEFFICIENTS

From paragraph III of DACO data sheet PMJ dated 3-6-57:

Nose gear damping force $P_o = K (\dot{S})^2$

Main gear damping force $P_o = 2.39 (\dot{S})^2$

$P_o \Longrightarrow$ strut load along strut (#)

$S \Longrightarrow$ strut stroke (in)

$S_{max} = 16.5$ in. fully compressed)

$K = 5.0 \quad 0 \leq S < 14.25$ in.

$K = 500.0 \quad 14.25 < S \leq 16.5$

In our system:

$h_1 = S_1 + H_{1o}, \; 1 = L, R, N$

$\therefore \dot{h}_1 = \dot{S}_1$

Let the damping contribution to landing gear ground reaction be:

$$\mathcal{T}_{Di} = - E_1 (\dot{h}_i)^2, \quad i = L, R, N$$

for $i = N$

$E = 5.0$

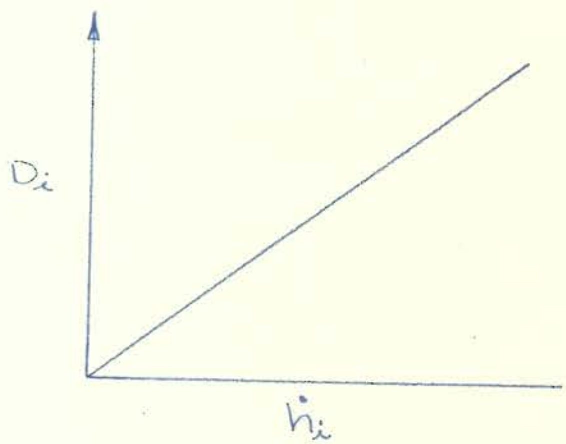
$$H_{N_o} \leq h_i < \left[H_{N_o} + 14.25\right] \quad \text{(in.)}$$

$E = 500.0$

$$\left[H_{N_O} + 14.25\right] < h_i \leq \left[H_{N_o} + 16.5\right] \quad \text{(in.)}$$

Let's gander at $E_i(\dot{h}_i)^2$.

Let $D_i = E_i\dot{h}_i$



$\therefore \tau_{D_i} = - D_i(\dot{h}_i)$

Other approach is to use:

$\tau_{D_i} = - E_i(\dot{h}_i)^2$

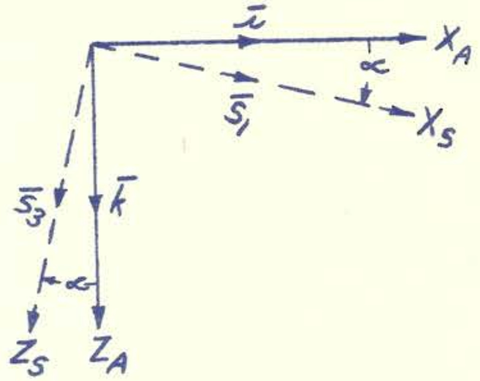
APPENDIX 9

Projection on Body Axes of Aerodynamic Moments Measured in Stability Axes

The aerodynamic moments as measured in the Stability Axes are

$$\bar{M}_S = (M_{xS})\bar{s}_1 + (M_{yS})\bar{s}_2 + (M_{zS})\bar{s}_3$$

Transfer equations stability to body system



$$\bar{s}_1 = \cos\alpha\,\bar{i} + \sin\alpha\,\bar{k}$$

$$\bar{s}_2 = \bar{j}$$

$$\bar{s}_3 = -\sin\alpha\,\bar{i} + \cos\alpha\,\bar{k}$$

$$\bar{M}_S = \begin{matrix} (M_{xS}\cos\alpha)\bar{i} & & + (M_{xS}\sin\alpha)\bar{k} \\ & + (M_{yS})\bar{j} & \\ (-M_{zS}\sin\alpha)\bar{i} & & + (M_{zS}\cos\alpha)\bar{k} \end{matrix}$$

$$\therefore \bar{M}_s = [M_{x_s} \cos\alpha - M_{z_s} \sin\alpha]\bar{i} + [M_{y_s}]\bar{j} + [M_{x_s} \sin\alpha + M_{z_s} \cos\alpha]\bar{k}$$

$$M_{x_s} = C_l \frac{\rho V_p^2}{2} S b$$

$$M_{y_s} = C_m \frac{\rho V_p^2}{2} S c$$

$$M_{z_s} = C_n \frac{\rho V_p^2}{2} S b$$

where b = wing span

c = either mean aerodynamic chord or mean geometric chord. Whichever applies will be specified in data defining C_m

$$\therefore \bar{M}_s = V_p^2 \frac{\rho S b}{2}[C_l \cos\alpha - C_n \sin\alpha]\bar{i} + V_p^2 \frac{\rho S c}{2}[C_m]\bar{j} + V_p^2 \frac{\rho S b}{2}[C_l \sin\alpha + C_n \cos\alpha]\bar{k}$$

APPENDIX 10

Moments about Body Axes Due to Aerodynamic Forces Measured in Stability Axes System

Note: These moments are in addition to those achieved by projecting onto the Body Axes aerodynamic moments as measured in the stability system.

Moments of Aero Forces about Body Axes

Let coordinates in body axes system of origin of stability axes system be represented by:

$$\bar{A} = (A_{x_A})\bar{i} + (A_{z_A})\bar{k} \; ,$$

$\bar{i}$ = unit vector X_a direction

$\bar{k}$ = unit vector Z_a direction

and the aero forces as measured in the stability axes system

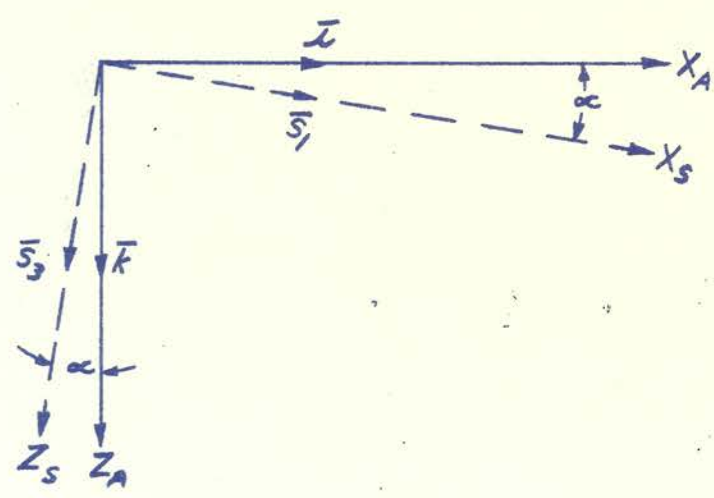
$$\bar{F} = (F_{x_S})\bar{s}_1 + (F_{y_S})\bar{s}_2 + (F_{z_S})\bar{s}_3$$

$\bar{s}_1$ = unit vector X_s direction

$\bar{s}_2$ = unit vector Y_s direction

$\bar{s}_3$ = unit vector Z_s direction

Transfer equations Stability to Body system



$$\bar{S}_1 = \cos\alpha\,\bar{i} + \sin\alpha\,\bar{k}$$

$$\bar{S}_2 = \bar{j}$$

$$\bar{S}_3 = -\sin\alpha\,\bar{i} + \cos\alpha\,\bar{k}$$

$$\therefore\quad \bar{F} = [F_{x_S}\cos\alpha]\bar{i} - [F_{Z_S}\sin\alpha]\bar{i} + [F_{y_S}]\bar{j} + [F_{x_S}\sin\alpha]\bar{k} + [F_{Z_S}\cos\alpha]\bar{k}$$

$$\bar{F} = [F_{x_S}\cos\alpha - F_{Z_S}\sin\alpha]\bar{i} + [F_{y_S}]\bar{j} + [F_{x_S}\sin\alpha + F_{Z_S}\cos\alpha]\bar{k}$$

$$\therefore \bar{M}_F = \bar{A}\times\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ (A_{x_A}) & 0 & (A_{Z_A}) \\ F_{x_A} & F_{y_A} & F_{Z_A} \end{vmatrix}$$

where
$$F_{x_A} = F_{x_S}\cos\alpha - F_{Z_S}\sin\alpha$$
$$F_{y_A} = F_{y_S}$$
$$F_{Z_A} = F_{x_S}\sin\alpha + F_{Z_S}\cos\alpha$$

$$\therefore \bar{M}_F = (-F_{y_A}A_{Z_A})\bar{i} + [F_{x_A}A_{Z_A} - F_{Z_A}A_{x_A}]\bar{j} + (F_{y_A}A_{x_A})\bar{k}$$

Note: These are moments <u>in addition</u> to the aerodynamic moments about stability axes and arise in the body due to non-coincidence of body system origin and stability system origin.

$$\overline{M_F} = \begin{Bmatrix} \left[-F_{y_S} A_{Z_A}\right]\overline{i} \\ +\left[\left(F_{x_S}\cos\alpha - F_{Z_S}\sin\alpha\right)A_{Z_A} - \left(F_{x_S}\sin\alpha + F_{Z_S}\cos\alpha\right)A_{x_A}\right]\overline{j} \\ +\left[F_{y_S} A_{x_A}\right]\overline{k} \end{Bmatrix}$$

From Appendix 5

$$F_{x_S} = -C_D \frac{\rho V_P^2}{2} S$$

$$F_{y_S} = C_y \frac{\rho V_P^2}{2} S$$

$$F_{Z_S} = -C_L \frac{\rho V_P^2}{2} S$$

$$\overline{M_F} = -V_P^2 \frac{\rho S}{2}\left[C_y A_{Z_A}\right]\overline{i}$$

$$+V_P^2 \frac{\rho S}{2}\left[\left(C_L\sin\alpha - C_D\cos\alpha\right)A_{Z_A} + \left(C_L\cos\alpha + C_D\sin\alpha\right)A_{x_A}\right]\overline{j}$$

$$+V_P^2 \frac{\rho S}{2}\left[C_y A_{x_A}\right]\overline{k}$$

APPENDIX 11

Moments about Body Axes due to Thrust

Assumption: Thrust vector known as components in body axes system

Let $\bar{B} = (B_{xA})\bar{i} + (B_{yA})\bar{j} + (B_{zA})\bar{k}$

represent point of application of thrust in body axes system

Let $\bar{T} = (T_{xA})\bar{i} + (T_{yA})\bar{j} + (T_{zA})\bar{k}$

represent thrust vector as measured in body axes system

∴ Thrust moments about body axes

$$\bar{M}_T = \bar{B} \times \bar{T} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ B_{xA} & B_{yA} & B_{zA} \\ T_{xA} & T_{yA} & T_{zA} \end{vmatrix}$$

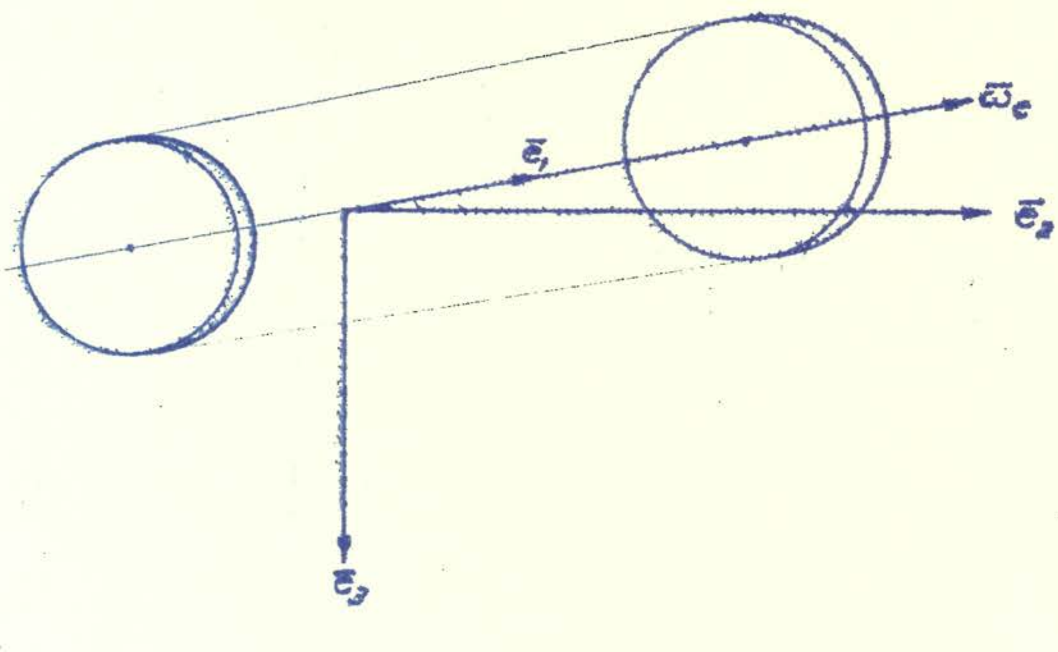
$$\bar{M}_T = \left[T_{zA} B_{yA} - T_{yA} B_{zA}\right]\bar{i} + \left[T_{xA} B_{zA} - T_{zA} B_{xA}\right]\bar{j} + \left[T_{yA} B_{xA} - T_{xA} B_{yA}\right]\bar{k}$$

APPENDIX 12

Engine Gyroscopic Moments Projected on Body Axes

Engine Gyroscopic Effect

$$\bar{H} = \sum m_e \left[\bar{r}_e \times (\bar{\omega}_e \times \bar{r}_e) \right]$$



Linear momentum about $\bar{e}_1$ axis

$$\bar{H}_e = \sum m_e \left[\bar{r}_e \times (\bar{\omega}_e \times \bar{r}_e)\right]$$

$$\bar{r}_e = r_1 \bar{e}_1 + r_2 \bar{e}_2 + r_3 \bar{e}_3$$

$$\bar{\omega}_e = \omega_e \bar{e}_1$$

$$(\bar{\omega}_e \times \bar{r}_e) = \begin{vmatrix} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \\ \omega_e & 0 & 0 \\ r_1 & r_2 & r_3 \end{vmatrix} = -r_3 \omega_e \bar{e}_2 + r_2 \omega_e \bar{e}_3$$

$$\bar{r}_e \times (\bar{\omega}_e \times \bar{r}_e) = \begin{vmatrix} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \\ r_1 & r_2 & r_3 \\ 0 & -r_3\omega_e & r_2\omega_e \end{vmatrix}$$

$$\bar{r}_e \times (\bar{\omega}_e \times \bar{r}_e) = \omega_e (r_2^2 + r_3^2) \bar{e}_1 - \omega_e r_1 r_2 \bar{e}_2 - \omega_e r_3 r_1 \bar{e}_3$$

$$\sum m_e \left[\bar{r}_e \times (\bar{\omega}_e \times \bar{r}_e)\right] = \sum m_e \left[\omega_e (r_2^2 + r_3^2) \bar{e}_1 - \omega_e r_1 r_2 \bar{e}_2 - \omega_e r_3 r_1 \bar{e}_3\right]$$

$$\sum m_e[\bar{r}_e \times (\bar{\omega}_e \times \bar{r}_e)] = [\omega_e \sum m_e(r_2^2 + r_3^2)]\bar{e}_1 - [\omega_e \sum m_e r_1 r_2]\bar{e}_2 - [\omega_e \sum m_e r_3 r_1]\bar{e}_3$$

The rotating parts of the engine are arranged symmetrically about the $\bar{e}_1$ axis

Because of the symmetrical distribution of mass about the $\bar{e}_1$ axis, for any mass particle of coordinate

$$\bar{r} = r_1\bar{e}_1 + r_2\bar{e}_2$$

there exists a symmetrically located mass particle with coordinates

$$\bar{r} = r_1\bar{e}_1 - r_2\bar{e}_2$$

Consequently, considering these two symmetrically located mass particles

$$\sum m_e r_1 r_2 = m_e r_1 r_2 + m_e r_1(-r_2) = 0$$

And since in general for any mass particle we can find another symmetrically located with respect to the $\bar{e}_1$ axis, in general then

$$\sum m_e r_1 r_2 = 0$$

and by similar application of the ancient and devious reasoning of alchemy

$$\sum m_e r_3 r_1 = 0$$

$$\therefore \bar{H}_e = [\omega_e \sum m_e(r_2^2 + r_3^2)]\bar{e}_1$$

$\sum m_e(r_2^2 + r_3^2) = I_e$ = moment of inertia of rotating engine parts about the engine axis of rotation

$$\bar{H}_e = \omega_e I_e \bar{e}_1$$

$$\frac{d}{dt}\bar{H}_e = I_e \dot{\omega}_e \bar{e}_1 + I_e \omega_e \frac{d\bar{e}_1}{dt} = \bar{K}_e$$

where $\bar{K}_e$ = external torque applied to engine

The reaction of the engine applied to its mounting is

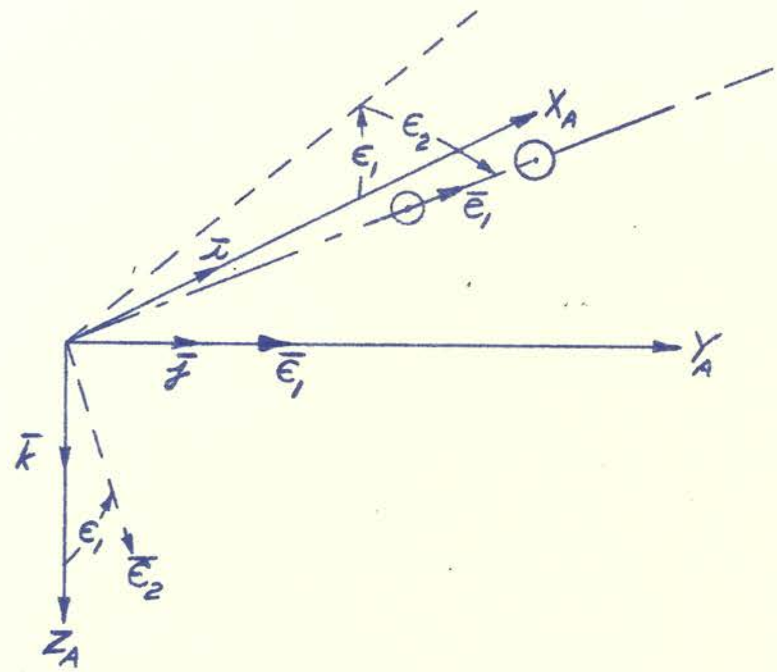
$$\bar{M}_e = -\bar{K}_e = -I_e \dot{\omega}_e \bar{e}_1 - I_e \omega_e \frac{d\bar{e}_1}{dt}$$

$$\frac{d\bar{e}_1}{dt} = \bar{\omega}_A \times \bar{e}_1$$

where $\bar{\omega}_A$ = rotational velocity vector of the aircraft

In general, the engine is canted with respect to the aircraft body axes system; a good example is engine mounting in the Martin P6M aircraft.

Let the engine orientation with respect to the body axes system be described by the two angles ϵ_1 and ϵ_2 as indicated in the following diagram.



$$\therefore \; \bar{e}_1 = [\cos\epsilon_2 \cos\epsilon_1]\bar{i} + [\sin\epsilon_2]\bar{j} - [\cos\epsilon_2 \sin\epsilon_1]\bar{k}$$

and since

$$\bar{\omega}_A = p_A\bar{i} + q_A\bar{j} + r_A\bar{k}$$

$$\frac{d\bar{e}_1}{dt} = [p_A\bar{i} + q_A\bar{j} + r_A\bar{k}] \times [(\cos\epsilon_2 \cos\epsilon_1)\bar{i} + (\sin\epsilon_2)\bar{j} - (\cos\epsilon_2 \sin\epsilon_1)\bar{k}]$$

$$\frac{d\bar{e}_1}{dt} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ p_A & q_A & r_A \\ \cos\epsilon_1 \cos\epsilon_2 & \sin\epsilon_2 & -\sin\epsilon_1 \cos\epsilon_2 \end{vmatrix}$$

$$\frac{d\bar{e}_1}{dt} = -\left[(\sin\epsilon_1 \cos\epsilon_2) q_A + (\sin\epsilon_2) r_A\right]\bar{i}$$
$$+\left[(\cos\epsilon_1 \cos\epsilon_2) r_A + (\sin\epsilon_1 \cos\epsilon_2) p_A\right]\bar{j}$$
$$+\left[(\sin\epsilon_2) p_A - (\cos\epsilon_1 \cos\epsilon_2) q_A\right]\bar{k}$$

$$\therefore \ \bar{M}_e = M_{e_{x_A}}\bar{i} + M_{e_{y_A}}\bar{j} + M_{e_{z_A}}\bar{k}$$

$$M_{e_{x_A}} = -I_e\dot{\omega}_e\left[\cos\epsilon_1 \cos\epsilon_2\right] + I_e\omega_e\left[(\sin\epsilon_1 \cos\epsilon_2) q_A + (\sin\epsilon_2) r_A\right]$$

$$M_{e_{y_A}} = -I_e\dot{\omega}_e\left[\sin\epsilon_2\right] - I_e\omega_e\left[(\cos\epsilon_1 \cos\epsilon_2) r_A + (\sin\epsilon_1 \cos\epsilon_2) p_A\right]$$

$$M_{e_{z_A}} = +I_e\dot{\omega}_e\left[\sin\epsilon_1 \cos\epsilon_2\right] - I_e\omega_e\left[(\sin\epsilon_2) p_A - (\cos\epsilon_1 \cos\epsilon_2) q_A\right]$$

DERIVATION

BODY AXES

EULER ANGLE RATE EQUATIONS

Prepared by: T. C. Denninger

March 5, 1957

Derivation of Body Axes Euler Angle Rate Equations

Definitions:

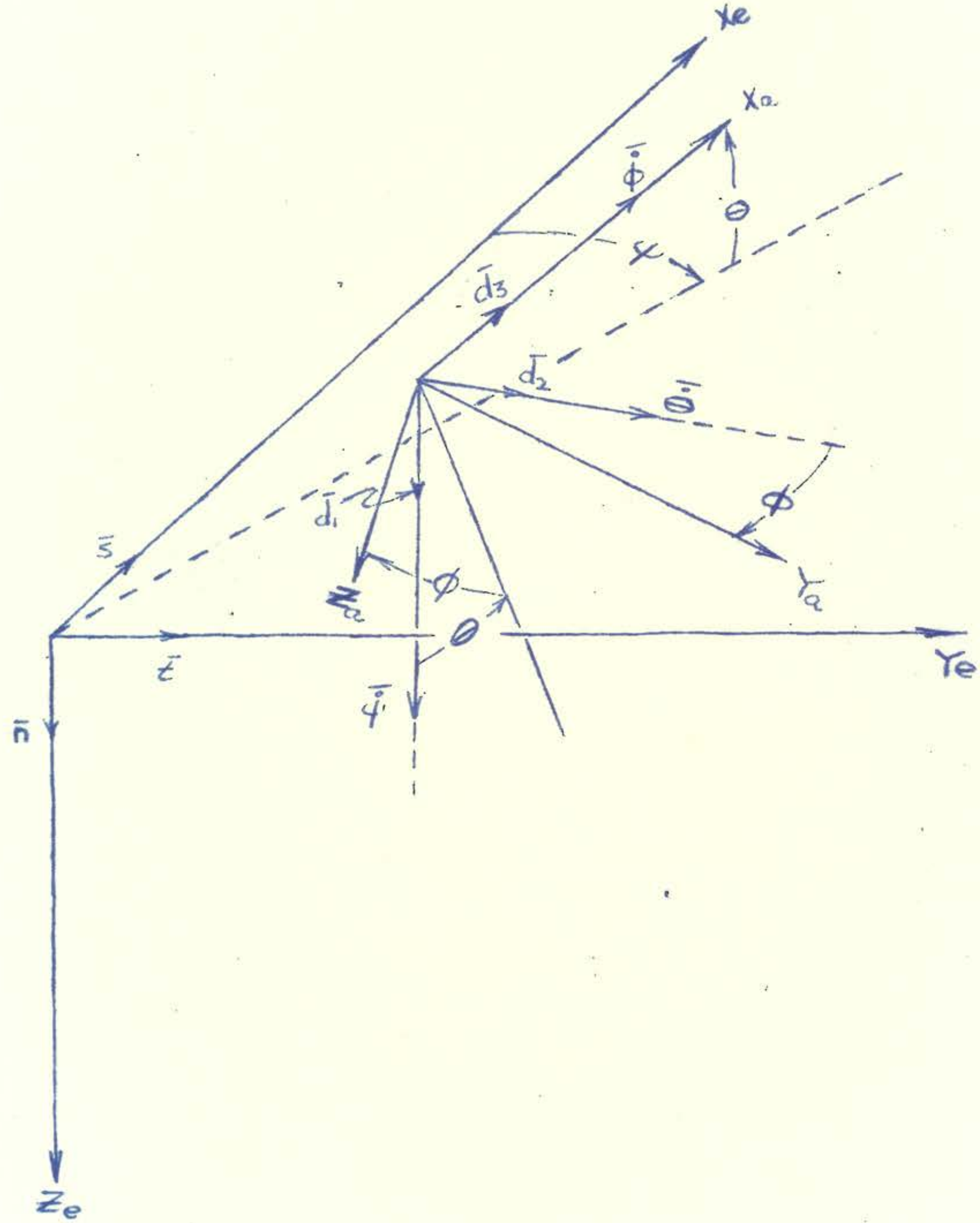
ψ = Body axes Euler heading angle; angle between the X_e inertial axis and the projection on the X_e, Y_e plane of the positive X_a body axis. Positive ψ is a clockwise rotation when looking the direction of the positive Z_e axis.

θ = Body axes Euler pitch angle; angle between the positive X_a body axis and the X_e, Y_e plane. θ is positive when the projection of the positive X_a body axis on the Z_e inertial axis is in the direction of the negative Z_e axis.

ϕ = Body axes Euler roll angle; angle between the positive Y_a body axis and that line in the Y_a, Z_a plane that is parallel to the X_e, Y_e plane and intersects the origin of the X_a, Y_a, Z_a body axes system.

For derivational purposes, the following sequence is associated with the body axes Euler angles:

1. With the body axes system initially parallel to the inertial axes system, rotate the body axes on angle ψ about the Z_a body axis.
2. With the body axes in the position attained by the previous step, rotate the body axes an angle θ about the Y_a body axis.
3. With the body axes in the position attained by the previous two steps, rotate the body axes an angle ϕ about the X_a body axis.


Xe
Xa
Ye
Ya
Ze
Za

Let $\bar{d}_1$, $\bar{d}_2$, $\bar{d}_3$ represent respectively unit vectors along the axes about which occur the rotations ψ, θ, ϕ. Using the right hand screw rule, let the unit vectors point in that direction in which positive ψ, θ, ϕ occur.

From step 1 in the Euler sequence

$$\bar{d}_1 = \bar{n}$$

From step 2

$$\bar{d}_2 = -\sin\psi\,\bar{s} + \cos\psi\,\bar{t}$$

From step 3

$$\bar{d}_3 = \cos\psi\cos\theta\,\bar{s} + \sin\psi\cos\theta\,\bar{t} - \sin\theta\,\bar{n}$$

Let the rates of change of the Euler angles be denoted by $\dot{\psi}$, $\dot{\theta}$, $\dot{\phi}$. Then in vector form

$$\bar{w} = \dot{\psi}\,\bar{d}_1 + \dot{\theta}\,\bar{d}_2 + \dot{\phi}\,\bar{d}_3$$

$$\overline{\omega} = \dot{\psi}\bar{d}_1 + \dot{\theta}\bar{d}_2 + \dot{\phi}\bar{d}_3$$

$$\bar{d}_1 = \bar{n}$$

$$\bar{d}_2 = -\sin\psi\,\bar{s} + \cos\psi\,\bar{t}$$

$$\bar{d}_3 = \cos\psi\cos\theta\,\bar{s} + \sin\psi\cos\theta\,\bar{t} - \sin\theta\,\bar{n}$$

$$\overline{\omega} = \left[-\dot{\theta}\sin\psi + \dot{\phi}\cos\psi\cos\theta\right]\bar{s} + \left[\dot{\theta}\cos\psi + \dot{\phi}\sin\psi\cos\theta\right]\bar{t}\left[\dot{\psi} - \dot{\phi}\sin\theta\right]\bar{n}$$

From X form equations appendix A4-5:

$$\overline{\omega} = \Big[(-\dot{\theta}\sin\psi + \dot{\phi}\cos\psi\cos\theta)\cos\psi\cos\theta + (\dot{\theta}\cos\psi + \dot{\phi}\sin\psi\cos\theta)\sin\psi\cos\theta$$
$$-(\dot{\psi} - \dot{\phi}\sin\theta)\sin\theta\Big]\bar{i}$$

$$\Big[(-\dot{\theta}\sin\psi + \dot{\phi}\cos\psi\cos\theta)(\cos\psi\sin\theta\sin\phi - \sin\psi\cos\phi) + (\dot{\theta}\cos\psi + \dot{\phi}\sin\psi\cos\theta)$$
$$(\sin\psi\sin\theta\sin\phi + \cos\psi\cos\phi) + (\dot{\psi} - \dot{\phi}\sin\theta)$$
$$\cos\theta\sin\phi\Big]\bar{j}$$

$$\big[(-\dot{\theta}\sin\psi + \dot{\phi}\cos\psi\cos\theta)(\cos\psi\sin\theta\cos\phi + \sin\psi\sin\phi) + (\dot{\theta}\cos\psi + \dot{\phi}\sin\psi\cos\theta)$$

$$(\sin\psi\sin\theta\cos\phi - \cos\psi\sin\phi) + (\dot{\psi} - \dot{\phi}\sin\theta)\cos\theta\cos\phi\big]\bar{k}$$

$$\bar{\omega} = p_a\bar{i} + q_a\bar{j} + r_a\bar{k}$$

$$p_a = (-\dot{\theta}\sin\psi + \dot{\phi}\cos\psi\cos\theta)\cos\psi\cos\theta + (\dot{\theta}\cos\psi + \dot{\phi}\sin\psi\cos\theta)\sin\psi\cos\theta$$

$$+(\dot{\phi}\sin\theta - \dot{\psi})\sin\theta$$

$$p_a = \dot{\phi}\cos^2\theta + \dot{\phi}\sin^2\theta - \dot{\psi}\sin\theta$$

$$p_a = \dot{\phi} - \dot{\psi}\sin\theta$$

$$q_a = (-\dot{\theta}\sin\psi + \dot{\phi}\cos\psi\cos\theta)(\cos\psi\sin\theta\sin\phi - \sin\psi\cos\phi) + (\dot{\theta}\cos\psi + \dot{\phi}\sin\psi\cos\theta)$$

$$(\sin\psi\sin\theta\sin\phi + \cos\psi\cos\phi) + (\dot{\psi} - \dot{\phi}\sin\theta)\cos\theta\sin\phi$$

$$q_a = (\dot\phi \cos\psi \cos\theta)(\cos\psi \sin\theta \sin\phi - \sin\psi \cos\phi) + (\dot\theta \sin\psi)(\sin\psi \cos\phi)$$
$$+ (\dot\phi \sin\psi \cos\theta)(\sin\psi \sin\theta \sin\phi + \cos\psi \cos\phi)$$
$$+ (\dot\theta \cos\psi)(\cos\psi \cos\phi) + (\dot\psi - \dot\phi \sin\theta) \cos\theta \sin\phi$$

$$q_a = \dot\phi \cos\theta \sin\phi + \dot\theta \cos\phi + \dot\psi \cos\theta \sin\phi - \dot\phi \cos\theta \sin\theta \sin\phi$$

$$q_a = \dot\psi \cos\theta \sin\phi + \dot\theta \cos\phi$$

$$r_a = (-\dot\theta \sin\psi + \dot\phi \cos\psi \cos\theta)(\cos\psi \sin\theta \cos\phi + \sin\psi \sin\phi)(\dot\theta \cos\psi + \dot\phi \sin\psi \cos\theta)$$
$$(\sin\psi \sin\theta \cos\phi - \cos\psi \sin\phi) + (\dot\psi - \dot\phi \sin\theta)(\cos\theta \cos\phi)$$

$$r_a = -\dot\theta \sin\phi + \dot\phi \cos\theta \sin\theta \cos\phi + \psi \cos\theta \cos\phi - \dot\phi \cos\theta \sin\theta \cos\phi$$

$$r_a = \dot{\psi} \cos\theta \cos\phi - \dot{\theta} \sin\phi$$

$$p_a = \dot{\phi} - \dot{\psi} \sin\theta$$

$$q_a = \dot{\psi} \cos\theta \sin\phi + \dot{\theta} \cos\phi$$

$$r_a = \dot{\psi} \cos\theta \cos\phi - \dot{\theta} \sin\phi$$

Rearranging the above equations:

$$\dot{\psi} = \frac{1}{\cos\theta} \left[q_a \sin\phi + r_a \cos\phi \right]$$

$$\dot{\theta} = q_a \cos\phi - r_a \sin\phi$$

$$\dot{\phi} = p_a + \dot{\psi} \sin\theta$$

A rapid review of what was done in deriving the body axes Euler angle rates: first the body axes rotational velocity vector was represented as the vector sum of three vectors.

$$\overline{\omega} = \dot{\psi}\,\overline{d}_1 + \dot{\theta}\,\overline{d}_2 + \dot{\phi}\,\overline{d}_3$$

Next, $\overline{\omega}$ was projected on the aircraft body axes system; the projections on the respective body axes being expressed as functions of $\dot{\psi}, \dot{\theta}, \dot{\phi}, \theta, \phi, \psi$. Then, since the aircraft rotational velocity vector can be expressed as

$$\overline{\omega} = p_a \overline{i} + q_a \overline{j} + r_a \overline{k},$$

the respective projections of $\overline{\omega}$ as functions of $\dot{\psi}, \dot{\theta}, \dot{\phi}, \theta, \phi$ were equated to p_a, q_a, r_a to obtain

$$p_a = \dot{\phi} - \dot{\psi} \sin\theta$$
$$q_a = \dot{\psi} \cos\theta \sin\phi + \dot{\theta} \cos\phi$$
$$r_a = \dot{\psi} \cos\theta \cos\phi - \dot{\theta} \sin\phi$$

These equations were then manipulated to obtain the equations for the body axes Euler angle rates.

Now its interesting to note that we can take the sums of the projections on the respective body axes of

$$\overline{\dot{\psi}} = \dot{\psi}\,\overline{d}_1$$
$$\overline{\dot{\theta}} = \dot{\theta}\,\overline{d}_2$$
$$\overline{\dot{\phi}} = \dot{\phi}\,\overline{d}_3$$

and equate those sums respectively to p_a, q_a, r_a but that we cannot do the inverse. In other words, the sums of the projections of

$$\overline{p}_a = p_a \overline{i}$$
$$\overline{q}_a = q_a \overline{j}$$
$$\overline{r}_a = r_a \overline{k}$$

in the respective $\bar{d}_1$, $\bar{d}_2$, $\bar{d}_3$ directions are NOT equal to $\dot{\psi}$, $\dot{\theta}$, $\dot{\phi}$. This is so because the vectors

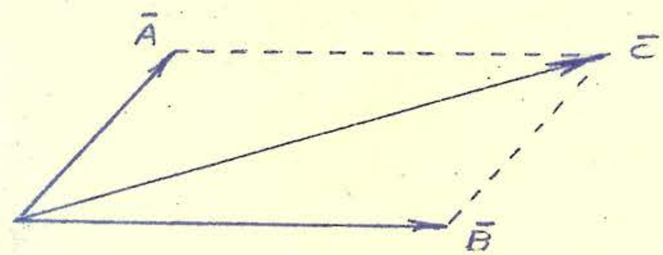
$$\bar{\dot{\psi}} = \dot{\psi}\,\bar{d}_1$$

$$\bar{\dot{\theta}} = \dot{\theta}\,d_2$$

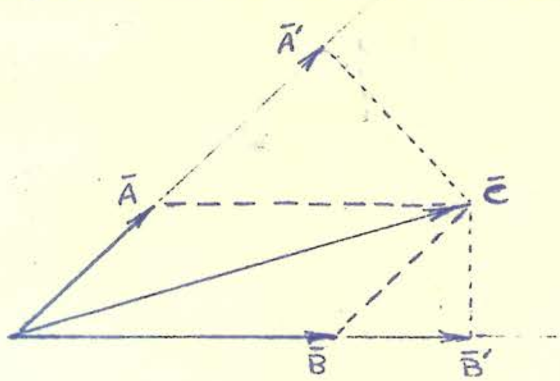
$$\bar{\dot{\phi}} = \dot{\phi}\,d_3$$

are not mutually orthoganol. $\bar{\dot{\psi}}$ and $\bar{\dot{\theta}}$ are perpendicular, $\bar{\dot{\psi}}$ is perpendicular to X_e, Y_e inertial system plane and $\dot{\theta}$ is contained in the X_e, Y_e plane, but $\bar{\dot{\phi}}$ is inclined to the X_e, Y_e plane by the angle θ and in general the angle θ is not zero. The reason why the non-orthogonality of the body axes Euler angles imposes a, so to speak, "irreversible" derivation can be seen by a simplified two dimensional example.

If we're given two non-orthogonal vectors $\bar{A}$ and $\bar{B}$ and construct the resultant $\bar{C}$ we get



If we can project $\bar{C}$ back on the lines of action of $\bar{A}$ and $\bar{B}$ we get



and it can be seen that by so doing we do NOT get back to the original vectors $\bar{A}$ and $\bar{B}$ since

$$\bar{A} \neq \bar{A}'$$
$$\bar{B} \neq \bar{B}'.$$

It is precisely this effect that would lead us to an erronous answer if we tried to directly derive $\dot{\psi}$, $\dot{\theta}$, $\dot{\phi}$ by projections of

$$\bar{p}_a = p_a \bar{i}$$

$$\bar{q}_a = q_a \bar{j}$$

$$r_a = r_a \bar{h}$$

in the $\bar{d}_1$, $\bar{d}_2$, $\bar{d}_3$ directions because in general $\bar{d}_1$, $\bar{d}_2$, $\bar{d}_3$ are NOT mutually orthoganal directions.

It is only when the original constituent vectors of a vector resultant are mutually perpendicular to each other that projecting the resultant vector on the lines of action of the constituent vectors gives identically the constituent vectors.

John Nash – 8 Aug

Some Noel &

About Guidance System. AGS

Project list – w/ key individuals

181 on BSG draftsman
x20 @ Edwards Air Force.
880 Commercial
Degree 62

Air Force Space Pgm

Space Pgm —

F111

LMS Reg long Pgm w/ Latin Prov.

Stabilization Control System.

Descent Engine – Steps

Integrate LMS & CMS
& ground support in Houston

FI 2 CMS
1 LMS

Houston 1 CMS
1 LMS

everything is ~~many~~ many relationships

in small launch window

CDRE – this not look at order
would this is space thing
elliptical mg ephemeris system
ΔT relationships

basically started on Air Force Space prm.

like a war things had to get done

Dick [illegible] Mgr Apollo

Frank Drake

Brian [illegible]